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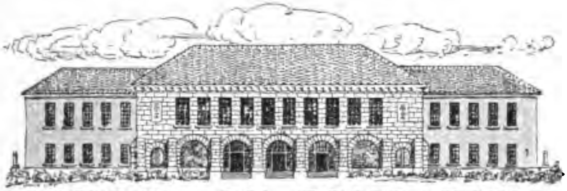

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PRACTICAL ALGEBRA

SECOND COURSE

BY

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COLLINS'S SECOND COURSE ALGEBRA.

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PREFACE

THE adoption of the plan of giving two courses in algebra, one early and for all students, and the other after one or two intervening years for those who specialize in science, is spreading. For the purposes of the latter a review is practically essential. This book consists of two parts, a *review* of the First Year Course, followed by *additional chapters* on more advanced topics, the whole providing the preparation for entrance to colleges of liberal arts. For a course covering the ground of both the First and Second Courses, the author's "Practical Elementary Algebra" is well adapted, though much that is in this Second Course is not in it.

Not a little has been written in this country within the past few years concerning the desirability of *unifying* the course in mathematics. A textbook for general use which aims to accomplish such an end must adapt itself to present-day conditions. Of the different branches of mathematics, the algebra easily provides the best means available of bringing together algebra, arithmetic, geometry, and applied mathematics, and this unification the present text undertakes to accomplish. In its preparation it was assumed that the pupil had taken the course usually offered in plane geometry. To be sure, the book can be used by those who have not taken geometry by simply omitting geometrical matter, whether text or exercises. In the same way, if so desired, the more purely arithmetical matter may be cut out, in part or in whole, since neither the arithmetical nor geometrical material conditions the subsequent algebraic matter.

Aside from preparation for the study of more advanced mathematics, probably the most directly practical thing learned from the study of algebra is the interpretation and use of *formulas*. Formulas introduced freely into the subject serve to vitalize it, since they connect it with practical problems. The attempt is made to train the student to thorough familiarity with the meaning and use of applied formulas.

The conception of functionality is developed as carefully and fully as it can be for pupils of the maturity of those who study algebra. The claims of this topic are now being strongly urged by the highest authorities.

Many of the features of the author's First Year Course, such as simplicity of presentation, minimum number of cases in topics, treatment of factoring and of graphs, constant reference to the axioms in the solution of equations, systematic testing for accuracy, translating to and from algebraic language, conservatism in the order of topics, etc., will be found likewise in this book. The inductive treatment, so prominent a feature of the First Course, is partly replaced in this by the deductive, because pupils are now far better prepared to understand its meaning.

The author is indebted to Professor E. S. Loomis of the West Side High School, Cleveland, Ohio, to Lewis Omer of the Academy at Evanston, Ill., to Miss Marie Gugle of the Central High School, Toledo, Ohio, and to Miss Theresa Moran of the High School, Stevens Point, Wis., for reading both the manuscript and the proofs and giving valuable advice. Professor H. L. Terry, State High School Inspector, Madison, Wis., read portions of the manuscript and offered helpful suggestions, as also did Professor A. L. Rhoton of Georgetown College, Ky. The following have read the proofs and given aid in other ways: President G. W. Brock, Alabama Normal College, Livingston, Ala.; Professor H. E. Cobb, Lewis Institute, Chicago; Professor Ira Condit, State Teachers' College, Cedar Falls, Iowa; Professor H. S. Hippensteel, State Normal School, Stevens Point, Wis.; and Professor T. M. Smith, Lash High School, Zanesville, Ohio.

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SUGGESTIONS TO TEACHERS

It is to be understood that the first 180 pages of this book constitute a review of the First Course. This part, containing a relatively small number of exercises, should be covered by classes much faster than the remaining portion. Different plans can be followed in passing over this part. Thus, one way is to start at the beginning and cover the whole ground rapidly, perhaps having the class do little more than read the text and model solutions and solve a few exercises here and there. Another way is to begin at factoring, or at equations, or at radicals, and then review the preceding topics as occasion requires. The numerous cross-references to earlier parts are intended to facilitate this plan. Classes usually differ greatly in preparation, even in the same school, and what would be the best course to pursue for one class might be a poor one for another.

The time allotted to advanced algebra varies greatly in different schools, and this, with varying preparation, makes it often necessary for the teacher to omit portions of the text not absolutely essential. If a course in algebraic transformations and manipulations as a preparation for college mathematics is the chief or sole aim of the study, then the more purely arithmetical and geometrical portions of the text can be omitted without injury to the continuity of the work. It is hoped, however, that these portions will be taught with the other, since they are calculated to strengthen the pupils' preparation in both algebra and mathematics in general, making the whole study much more practical. Suggestions are offered in various places for transposing the order of chapters and topics.

PRACTICAL ALGEBRA

CHAPTER I

THE FUNDAMENTAL OPERATIONS

I. THE ARABIC* AND ALGEBRAIC NOTATIONS

1. Elementary Algebra may be described as that branch of mathematics which employs both letters and figures in the study of numbers, and recognizes two kinds of numbers, positive and negative. Algebra is also often described as the study of the equation.

2. A **number** is one or more units (or ones). It is altogether likely that the original idea of numbers included only integers. At first a distinction was made only between one and several in a vague way. Later *counting* was introduced. Counting consists in having a reference series of objects, such as the fingers, or names to take their places, and pointing to the several objects counted, meanwhile saying the names of the standard series. The last name is the name of the number of objects counted. Later the idea of measurement was connected with the conception of numbers. Thus, to measure the length of a board, one applies his foot rule or meter stick a number of times. The number of times the measuring unit is moved, counted in the scale just described, is the number. Evidently from the standpoint of measurement number is always abstract, being a quotient of one magnitude divided by another of the same kind. Pupils are prone to fail to distinguish between abstract number itself, and the word, or figures, or letter, that denotes it.

A system for denoting numbers and making numerical calculations is called a **notation.**

---

* In all probability the so-called Arabic notation was invented in India in the period from the second to the seventh centuries A.D. It was transmitted to European nations through the Arabs, and so has been known for centuries as the Arabic notation for numbers.

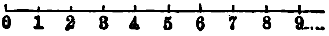
3. The Arabic Notation uses figures to denote the first nine numbers and zero, recognizes *place* value, employs absence of sign between figures to indicate addition, and is constructed on a scale of *ten*.

Thus, in 325, the 2 gets its meaning of 2 *tens* from its *place* to the left of 5, and the 3 its meaning of 3 *hundreds* from its place. Also, because juxtaposition (that is, placing figures next to each other with no sign between) means addition, 325 means $300 + 20 + 5$.

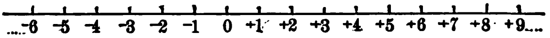
4. The Algebraic Notation uses both figures and letters to denote numbers, the letters denoting *any* numbers, but the absence of sign between letters, or between a figure and a letter, or between a figure or a letter and a parenthesis, or between two parentheses, indicates *multiplication*. For other operations special symbols are employed, as + to denote addition, –, subtraction, and so on.

Thus, $3ab + c$ denotes the number c added to the product of 3, a, and b. The algebraic notation also marks one series of numbers extending on one side of 0 with a + sign, and those on the other side of 0 extending in the opposite direction with a – sign.

Arithmetic Series:



Algebra Series:



As regards concrete applications the arithmetic series has far the wider use. The algebra series is best adapted to solving special kinds of problems which involve the presence of two kinds of numbers opposite in sense, as east and west longitude, debts and credits, numbering streets east and west or north and south, etc. But algebra *can* be used equally well in the solution of equational problems whose concrete numbers are arithmetical in character.

5. A Universal Language. The Arabic and Algebraical notations are in general use among the various civilized nations, being practically the same in different languages, though minor differences exist. Thus, in the United States and Great Britain, both ÷

and : are used to denote division, while on the continent of Europe only : is so employed.

6. Economy of Notation. Both the Arabic and the Literal notation are wonderfully abridged and compact as compared with ordinary writing in words. Thus, the number 878, which has only three figures, written out in words calls for twenty-four characters, and the formula $(a+b)^2=a^2+2ab+b^2$, translated into words, gives rise to nine times as many marks or characters.

7. Classes of Symbols in Algebra. Algebraic symbols are classified as follows:

(1) Symbols of *number*, as letters and figures.

(2) Symbols of *operations*, as $+$, $-$, $\pm$, $\times$ or $\cdot$, $\div$, $:$, $\surd$, and exponents.

(3) Symbols of *aggregation*, as $(\,)$, $[\,]$, $\{\,\}$, $\overline{\qquad}$.

(4) Symbols of *relation*, or relative size, as $=$, $\equiv$, $>$, $<$, $\not>$, $\not<$.

The angles ($>$, $<$) are called symbols of inequality, the opening being towards the larger quantity. Lines across the symbols of relation deny what the symbols unmarked signify. Thus, $\neq$ means "is not equal to"; $\not>$ means "is not greater than."

(5) Symbols of logic, as $\therefore$, meaning therefore, hence; and $\because$, meaning since, because.

(6) Symbols of continuation or omission, as 1, 2, 3, $\cdots$; see § 4.

8. The Symbols + and − are commonly considered to have two meanings:

(1) That of addition or subtraction, as $3+4=7$; $6-2=4$.

(2) That of quality, as $+5$, meaning 5 from the positive series, and -3, meaning 3 from the negative series. When the signs have the latter meaning they are usually put along with the quantity to which they are attached inside parentheses.

Thus, $(-6)-(+4)$ means 4 from the positive series is to be subtracted from 6 of the negative series.

9. The Symbols 0, 1, and ∞ are noteworthy. The first, 0, has a place in the ordinal series $\cdots -3, -2, -1, 0, +1, +2, +3, \cdots$.

It may also be regarded as belonging to the cardinal numbers. It has the following properties: $a-a=0$; $0\times a=a\times 0=0$. Division by 0 has no meaning.

The symbol 1 has the following properties: $a\div a=1$; $a\times 1=a\div 1=a$.

The symbol ∞ (called **infinity**) denotes an exceedingly large number, greater than any that can be named, and continually increasing.

II. DEFINITIONS

10. A **quantity** in algebra is a *number.* This number may be expressed by a single letter, or by a figure, or by a combination of figures, letters, and signs. Thus, 8, a, and $4a-3b+5c$ are each a quantity. "Quantity" and "algebraic expression" are used synonymously.

11. An **exponent** is a number written to the right of and higher than a quantity. When it is a positive whole number, it shows how many times the quantity is to be taken as a factor.

Thus, $4^3=4\times 4\times 4$; $a^4=a\times a\times a\times a$.

12. A **power of a number** is the continued *product* obtained by using 1 and the number one or more times as factors.

Thus, $1\times a\times a$, or a^2, is called the *square* of a; and $1\times a\times a\times a$, or a^3, is called the *cube* of a; $1\times x\times x\times x\times x$, or x^4, is called the *fourth power* of x; and so on.

13. A **root of a number** is one of its equal factors.

Thus, $\sqrt[3]{64}=4$; $\sqrt[5]{32}=2$. In $\sqrt[3]{64}$, 3 is called the *index* of the root. It shows into how many equal factors the number 64 is to be separated. When no index is written, 2 is understood. Thus, $\sqrt{9}=3$. The symbol $\surd$ is called the radical sign.

14. A **term,** or **monomial,** is an algebraical expression, or quantity, whose parts are not separated by a $+$ or $-$ sign.

Thus, 3, a, $5bc$, $12a^2bc$ are each a term, while the quantity $15m^2-3n$ consists of two terms. But in $3(a-b)+2(x+y)$, $3(a-b)$ is commonly regarded as a term, and $2(x+y)$ as one, the parentheses quantities being treated as single letters.

15. A **polynomial** is a quantity consisting of two or more terms. A quantity consisting of two terms is a **binomial**, one of three terms is a **trinomial**, one of four terms a **quadrinomial**, one of five or more terms a **multinomial**.

By a "**polynomial in x**" is meant an expression containing different powers of x; as, $6x^5 + 4x^4 - 11x^3 + x^2 - 7$.

16. The factors of a term are the **coefficient** and the **literal (letter) part**. Thus, in $5axy$, 5 would be regarded as the coefficient (or "with factor"). In ax^2y, 1 can be regarded as the coefficient of ax^2y, or, remembering the meaning of "coefficient," a can be regarded as the coefficient of x^2y, or even x^2 as the coefficient of ay, if so desired. When no coefficient or exponent is written, 1 is understood. Thus, $a = 1a^1$.

The student should note carefully the difference between a coefficient and an exponent. Thus, $4x = x + x + x + x$, while $x^4 = x \times x \times x \times x$; $4x$ means 4 times x, while x^4 is shorthand for $xxxx$. One should think of four x's multiplied together when he sees x^4.

17. **Similar terms** are those containing the same letters, the corresponding letters having the same exponents. They can differ in signs and coefficients.

Thus, $9x^2y^3$ and $-13x^2y^3$ are similar terms. Similar terms can be combined into a single term. Thus, $6a^2b + 10a^2b = 16a^2b$.

18. **The Degree of a Term** is the sum of the exponents of its literal factors. For example, $9m^3n^2p$ is of the sixth degree because $3 + 2 + 1 = 6$; but $3abx^3y$ is of only the fourth degree if we think of $3ab$ as the coefficient of the unknown factors x^3 and y.

19. **Homogeneous Quantities** are those whose terms are all of the same degree. Many expressions in algebra are homogeneous, and errors in handling such expressions can be detected by applying the test of homogeneity.

Thus, $a^3 + 3a^2b + 3ab + b^3$ for $(a + b)^3$ is *wrong*, since all the terms should be of the third degree, and $3ab$ is of only the second degree; this term should be $3ab^2$.

III. ORDER OF PRECEDENCE OF OPERATIONS

20. The **numerical value** of a quantity is found by assigning certain values to the letters contained in it, and simplifying the result.

Thus, the numerical value of $2a - 3m$ when $a = 5, m = 2$, is 4.

21. Order of Precedence of Operations. The order of performing operations in algebra is determined by certain rules.

1. *In any term* (§ 14), *symbols of aggregation being absent, raising to powers and extracting roots must be performed* **before** *multiplications and divisions.*

Thus, $2 \times 3^2 = 18$; $3x^2 = 3xx$; while $(3x)^2 = 9xx$.

2. *Symbols of aggregation being absent, multiplications and divisions must be performed* **before** *additions and subtractions.*

Thus, $3 + 4 \times 5 = 23$; $5 \times 2^2 - 2^3 \div 4 = 18$.

3. *Operations inside symbols of aggregation are performed before those outside.*

Thus, $8(3 + 4) = 8 \times 7 = 56$. $(2 \times 3)^2 = 6^2 = 36$.

EXAMPLE. If $a = 3$, $b = 2$, $c = 64$,

$$15ab^2 - 3b^3\sqrt[3]{c}$$

becomes $15 \times 3 \times 2^2 - 3 \times 2^3 \times \sqrt[3]{64}$ (By § 20)

or, $15 \times 3 \times 4 - 3 \times 8 \times 4$, or 84. (By §§ 12, 13)

a. Multiplications and divisions denoted by ×, ·, ÷ are performed in order from left to right, but when multiplications are denoted by juxtaposition, as in $4c \div 3ab$, the multiplications in dividend and divisor must be performed first. Hence, when translating this notation to that of ×, ·, ÷ parentheses must be introduced.

Thus, if $a = 12$, $b = 3$, $c = 4$.

$a \div b \times c = 12 \div 3 \times 4 = 16$; while $a \div bc = 12 \div (3 \times 4) = 1$.

22. Importance of the Rules of Precedence of Operations. If strict attention is not paid to the order in which operations are to be performed, great confusion will result.

Thus, $(2+3)\times 4^2=80$; while $2+(3\times 4)^2=146$; and $(2+3\times 4)^2=196$; and $2+3\times 4^2=50$, by the rules of § 21.

Some other rules than those just described *might* have been adopted by those who introduced the algebraic notation, but the rules of § 21 represent careful testing and are well adapted to the purpose intended. Those who study or use algebra must make their calculations conform to the laws of precedence of operations.

23. Exercise in simplifying Expressions and finding Values.

1. $3+7-6+4-2$.
2. $3+6\div 2$.
3. $4(6-3)-2$.
4. $8\times 6\div 3-12$.
5. $24\div(2\times 3)$.
6. $(5-2)(11-3\times 2)$.
7. $2+16\div 2$.
8. $2\cdot 7\cdot 5-3^2\cdot 4$.
9. $2(13+7^2-41)-15$.
10. $\sqrt{[3(19+50-42)]}+26$.

24. Exercise in translating Algebraic Expressions into Ordinary Language. To change algebraic language into ordinary words is, in many respects, like translating from one language to another, as English to German, or Latin to English. The student will find that quantities which take many words and much circumlocution for their description in ordinary language are expressed with very few symbols and very quickly in the algebraic notation. For this cause we can work with and reason about quantities written in the algebraic notation easily, when dealing with the same quantities in ordinary language would be almost out of the question on account of the complexity of the language. The great mathematician Fourier said of the algebraic notation, "There can be no more universal or more simple language, no language more exempt from error and obscurity."

1. Translate $4ab^2-3a^3\sqrt{c}+9c$ into ordinary language, and then find its numerical value when $a=2$, $b=3$, $c=4$.

TRANSLATION. "From 4 times a times b times b take 3 times a times a times a times the square root of c, and add 9 times c to the remainder."

To evaluate $4ab^2 - 3a^3\sqrt{c} + 9c$,

we have $4 \times 2 \times 3 \times 3 - 3 \times 2 \times 2 \times 2 \times \sqrt{4} + 9 \times 4$,

that is, $72 - 48 + 36$, or 60. *Ans.*

2. Translate $3x^3 + 2xy^2 - y\sqrt{z}$, and evaluate when $x = 3$, $y = 1$, $z = 9$.

3. Translate $3x(y^2 - z)$, and evaluate when $x = 5$, $y = 8$, $z = 25$.

Suggestion. "3 times x times the quantity which is the difference between y times y and z."

a. It is convenient to use the phrase "the quantity," or "the binomial," etc., to indicate what is included in parentheses.

Translate the following 15 exercises and evaluate them when $x = 3$, $y = 5$, $z = 4$:

4. $3x(y - z)$.
5. $4(xy - z)$.
6. $3x^2 + 2y^2$.
7. $9x^2yz - 2y^3$.
8. $3xy^2 - 2z\sqrt{z}$.
9. $4(x^2 + y^2 - z^2)$.
10. $3(xy^2z)^2$.
11. $4(xy^2 - x^2y)$.
12. $2(3xy^2 - z)^2$.
13. $(2x - y)^2z$.
14. $2\sqrt{y^2 - x^2}$.
15. $(x + 2y)(3y - z)$.
16. $4(2xy)^4z$.
17. $\sqrt{3 \cdot 2y^xz}$.
18. $12z^x + x^xy^x$.

25. Translation of Formulas into Ordinary Language.

1. If a denotes the area of a rectangle, b the length of its base, and h its height or altitude, translate the formula $a = bh$ into words.

Ans. "The number of square units in the area of a rectangle equals the product of the number of linear units in its base by the number of linear units in its altitude."

2. If c represents the total cost of n articles and p the price of one article, translate $c = np$ into words.

3. If V represents the volume of a rectangular box, l its length, w its width, and d its depth, translate $V = lwd$ into words. If S represents its surface, translate $S = 2(lw + ld + wd)$ into words.

4. If A denotes the area of a triangle, b its base, and h its altitude, translate $A = \frac{1}{2}bh$ into words.

5. If a represents the area of a circle, and r its radius, translate $a = \pi r^2$ into words.

6. If a represents the area of a trapezoid, b and b' its bases, h its altitude, translate $a = \frac{1}{2}(b + b')h$ into words.

7. If a represents the area of a rectangle, l its length, and b its breadth, express in words $a = bl$, $b = \frac{a}{l}$, and $l = \frac{a}{b}$ respectively.

8. If s represents the distance through which a body moves, v its uniform velocity or speed, and t the number of seconds it is in motion, what does $s = vt$ say?

9. If l denotes the length of a rectangle, and w its width, express in words $l = 2w + 3$; also $lw = 25$.

10. If $c =$ centigrade reading, or number of degrees on centigrade thermometer, and $f =$ reading at same time on common Fahrenheit thermometer, translate the formula, $c = \frac{5}{9}(f - 32°)$, into words.

26. Exercise in translating from Ordinary Language to that of Algebra.

1. The difference between the squares of a and b. *Ans.* $a^2 - b^2$.

2. The sum of the squares of a, b, c, and d.

3. The product of the sum of the squares and the difference of the squares of x and y.

4. The quotient of the difference between the cubes of a and x divided by the difference of their squares.

5. The difference between the product of the three factors, the square of a, b, and y, and the product of 3 and the square of x.

6. If p represents the price of eggs, n the number of dozen bought, and c the cost, state in algebraic language that the cost is the product of the price multiplied by the number of dozen bought.

27. Reading of Algebraic Expressions in Practice. While it is very important that the student should understand thoroughly the translation of expressions from algebraic language to English words, and conversely, in practice the algebraic expressions are read very much as they are written. It is customary when one comes to a polynomial in parentheses to say "the quantity" so and so. Thus, $3a+5(c-d)$ would be read "three a plus five times the quantity c minus d." It is more explicit, however, to say "the binomial," "trinomial," etc., instead of "the quantity."

The student may be interested in comparing the number of characters (letters, signs, and marks of all kinds) needed to express quantities and relations in the two notations. It will be found that as a rule it takes something like ten times as many characters in ordinary language as in the algebraic notation. Stenography, or shorthand, requires more marks than algebra.

IV. ADDITION AND SUBTRACTION

28. Addition in algebra means literally *putting together.* It differs from arithmetical addition in that both positive and negative numbers are united into one *sum,* while arithmetic adds unsigned numbers only. Thus, in algebra, debts and credits, or motions forward and backward on a straight line, can be added. In such additions positive numbers are written preceded by the sign +, or no sign, and negative numbers by −, the numbers with their signs being inclosed in parentheses. See § 8.

29. Assumed Laws Governing Addition.

I. ADDITION IS COMMUTATIVE, *that is, the order of the terms added can be changed in any way and the sum will not be altered provided each term retains its own sign.*

For example, $a+b-c=b-c+a=a-c+b$.

Evidently the sum of a set of credits and debts is the same no matter in what order they are taken.

II. ADDITION IS ASSOCIATIVE, *that is, terms can be grouped in any way by inclosing them in parentheses preceded by +, and the sum will not be changed, provided each term retains its own sign.*

Thus $(a+b)+c=a+(b+c)$.

It is clear that a set of credits and debts placed on separate sheets of paper and these arranged in any order can be *grouped* in any way with rubber bands without altering the sum.

30. Rule for Addition. The rule commonly followed in algebra is that used by the bookkeeper when he adds his debts and credits separately, and then combines the results.

31. Exercise in Addition.

1. Find the sum of \$3, – \$6, – \$11, \$22, and – \$5.

STATEMENT. $3 + (-6) + (-11) + (+22) + (-5)$.

SOLUTION. $3+22=25$; $(-6)+(-11)+(-5)=-22$; $25+(-22)=3$. *Ans.*

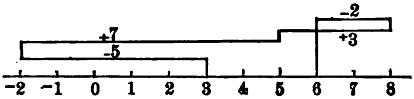
CHECK. $3+(-6)=-3$; $(-3)+(-11)=-14$; $(-14)+(+22)=+8$; $(+8)+(-5)=+3$. *Ans.*

Solve and check the following seven exercises orally:

2. $5\text{¢} + (-3\text{¢}) + (-18\text{¢}) + (+12\text{¢}) + (-2\text{¢}) = ?$

3. Sums of money deposited in a bank being marked + and other sums checked out marked –, find how much money in the bank a man has whose book shows the following items: + \$150, – \$10, – \$25, + \$30, – \$17, – \$25, + \$40.

4. Find the sum of the following lines whose several lengths are $+3$, -5, $+7$, $+3$, -2. *Ans.* $+6$.



5. Find the sum of the lines $+4$, -7, -3, $+2$, $+10$, -5, by a diagram, checking with ordinary solution.

6. If eastward motion is marked + and westward –, find the sum of the following movements of a football: + 61 ft., – 19 ft., + 33 ft., – 57 ft., + 10 ft., – 21 ft., + 75 ft.

7. Add $9a^2$, $-7a^2$, $13a^2$, $-25a^2$, $-11a^2$.

8. Add $13mn$, $-7mn$, $16mn$, $-11mn$, $-2mn$, $-mn$.

9. Add $12a^2b$, $-14a^2b$, $-10a^2b$, $15a^2b$, $6a^2b$, writing the quantities for the addition in a vertical column.

10. Add $3a^2b$ and $3ab^2$. *Ans.* $3a^2b + 3ab^2$. Why?

11. Add $4a^2x - 6a^2y + 9y^2$, $5a^2x - 14a^2y + 17y^2$, and $-2a^2x + a^2y - 11y^3$, setting the problem down to perform the addition so that similar terms fall in the same columns. Check with $a = 2$, $x = 3$, $y = 1$.

SOLUTION

$$\begin{array}{rrrrcrrrrcr}
4a^2x & -\ 6a^2y & +\ 9y^2 & & = & 48 & -24 & +\ 9 & & = & 33 \\
5a^2x & -\ 14a^2y & +\ 17y^2 & & = & 60 & -56 & +17 & & = & 21 \\
-2a^2x & +\ a^2y & & -\ 11y^3 & = & -24 & +\ 4 & & -\ 11 & = & -31 \\
\hline
7a^2x & -\ 19a^2y & +\ 26y^2 & -\ 11y^3 & = & 84 & -76 & +26 & -\ 11 & = & 23
\end{array}$$

12. Find the sum of $3x^2y^2 - 10y^4$, $-x^2y^2 + 5y^4$, $8x^2y^2 - 6y^4$, and $4x^2y^2 + 2y^4$. Check with $x = 2$, $y = 1$.

13. Add $\frac{1}{8}x^4 - \frac{1}{5}x^3 - \frac{1}{2}x^2 + 6$, $\frac{5}{6}x^4 + \frac{1}{6}x^3 - \frac{1}{12}x$, $-\frac{3}{4}x^4 - \frac{2}{15}x^3 + \frac{1}{5}x^2 - \frac{1}{4}x + 1$, and $\frac{1}{12}x^4 - 1\frac{7}{15}x^3 - 2x^2 + 3\frac{1}{2}x - \frac{1}{5}$.

32. Subtraction in algebra is the process of finding a quantity (called the *difference* or *remainder*) which added to one of two given quantities (the *subtrahend*) produces the other (the *minuend*).

Thus, \$6 − \$4 = \$2; 4¢ − (−3¢) = +7¢; $-5-(+4)=-9$; because in each case the algebraic sum (§ 31) of the remainder and subtrahend equals the minuend.

Subtraction in algebra can be defined in another way by referring to the algebraic series of § 4. Thus, the difference of two algebraic numbers is always the *distance* between them as they stand in the scale extending in opposite directions from 0; the difference is marked + or − according as the minuend is in the positive direction from the subtrahend or the negative direction.

As, \$7 − \$5 = \$2; 4¢ − (−3¢) = +7¢; $-5-(+4)=-9$.

Evidently both definitions give the same results. The second definition is more satisfactory as an *explanation of the nature of subtraction.*

33. Two Rules for Subtraction can be given, the second of which is somewhat artificial, though generally used. Both rules always give the same result.

1. *To subtract, find a number which added algebraically to the subtrahend gives the minuend.*

2. *Conceive the sign of the subtrahend to be changed, and add the result algebraically to the minuend.*

It is highly important that the student learn to be accurate in subtraction.

34. Exercise in Subtraction. Find the remainder in the following problems, being careful to give the proper sign to the answer. The solutions should be performed mentally to the 14th. The answers can be checked by testing the result by the scale.

1. $12-(-6)$ 2. $-120-(-60)$ 3. $-16-(+25)$

4. $\begin{array}{r} +24 \\ \underline{+38} \end{array}$ 5. $\begin{array}{r} -1400 \\ \underline{-3200} \end{array}$ 6. $\begin{array}{r} 4\,xyz \\ \underline{15\,xyz} \end{array}$

7. $\begin{array}{r} 6\,a^2b \\ \underline{-9\,a^2b} \end{array}$ 8. $\begin{array}{r} -3\,abc \\ \underline{11\,abc} \end{array}$ 9. $\begin{array}{r} -3\,a^2 \\ \underline{-12\,a^2} \end{array}$

10. Find the difference in fortunes of two men, one worth \$2500, and the other \$1400 in debt, *i.e.* find the difference between \$2500 and $-$ \$1400.

11. Find the difference in longitude between Alexandria, Egypt, 30° E. (+ 30° say), and New Orleans, 90° W. (–).

a. The first named of two quantities in a problem in subtraction is to be understood as the minuend. The answer, + 120°, shows that Alexandria, the first named, is east (+) of New Orleans, the subtrahend.

12. How many blocks apart are 2401 (each hundred means a block, and the last two figures give the house in the block) North Chestnut St. and 1401 South Chestnut St. if north is taken +?

13. What is the difference between the greatest recorded extremes of temperature at St. Paul, Minn., of $+104°$ and $-41°$?

14. From $3\,a^2b-2\,bc+6$ take $2\,a^2b-7\,bc-3$, writing the subtrahend under the minuend with similar terms (§ 17) in the same columns to perform the subtraction. Check with $a=2$, $b=3$, $c=2$.

15. From $18\,a^{2m} - 16\,a^m b^n - 25\,b^{2n}$ take $14\,a^{2m} - 24\,a^m b^n + 11\,b^{2n}$.

16. From $1\frac{1}{5}\,x^3 - 7\frac{1}{2}\,x^2 - \frac{5}{4}x + 11\frac{3}{5}$ take $\frac{7}{8}x^3 - 2\frac{1}{12}\,x^2 + 1\frac{5}{14}\,x - 7\frac{3}{8}$.

17. From $-x^4 - x^2 - x$ subtract $x^3 - 1$.

18. From the sum of 6×2^3, 2^3, and 3×2^3 take -11×2^3.

35. Addition and Subtraction in Algebra. Addition of Compound Terms. In algebra problems in subtraction, or combined additions and subtractions, can be changed into problems in addition, or can be regarded as problems in addition even when not so changed. Thus, $6 - 4$; *i.e.* $6 - (+4) = 6 + (-4)$, since each equals 2.

Similarly, $6 + 12 - 13 - 7 + 4 - 12 = ?$
becomes $6 + 12 + (-13) + (-7) + 4 + (-12) = ?$
which is now a problem in addition.

Add mentally in the following four problems, checking the result by adding seriatim, as in § 31:

1. \$12 + \$9 − \$19 + \$2 − \$13. 2. $14 - 11 - 25 - 3 + 11 + 6$.

3. Add $3(a - b)$, $5(a - b)$, and $-2(a - b)$.

4. From $12\,a(a^2 - 2\,b - c)$ take $-4\,a(a^2 - 2\,b - c)$.

5. In a merchant's cash register at the end of a day's business there were ten \$5 bills, twelve \$2 bills, twenty \$1 bills, four checks in his favor for \$12.10, \$8.25, \$10, \$25 respectively, \$18.46 in silver, a note in his favor for \$28.25, and a record on slips of paper of two notes against him, one for \$17.28 and another for \$60. His bank book showed he had \$228.68 in the bank, but during the day he had issued four checks for \$9.10, \$7.20, \$1.17, and \$5 respectively, and had received three due bills, one for \$22, the second for \$9.72, and the third for \$3.49. What will the merchant get for the *sum* if he adds all these items?

6. From $a^4 - b^4$ take $4\,a^3b - 6\,a^2b^2 + 4\,ab^3$ and from the result take $2\,a^4 - 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 - 2\,b^4$.

7. If $A = 3\,x^2 + 2\,x - 3$, $B = x^3 - 2\,x^2 + 2\,x + 1$, $C = x^2 - 6\,x + 5$, find the value of (1) $A + B + C$, (2) $A + B - C$, (3) $A - B - C$.

8. What must be added to $r^2+s^2+t^2$ to produce $3r^2-s^2$?

9. Add $2(x+y)+3(x-y)$ and $3(x+y)-2(x-y)$.

10. Add $4(a+2b)^2-4(a-m)$, $-20(a+2b)^2+5(a-m)$, $12(a+2b)^2-14(a-m)$, and $-5(a+2b)^2+5(a-m)$.

11. Add ax, bx, and cx, with respect to x. *Ans.* $(a+b+c)x$.

SUGGESTION. These terms evidently are added just as one would add $3x$, $5x$, and $7x$.

12. Add $ax+by$, $cx+dy$, $ex+fy$, and $gx+hy$ with respect to x and y.

13. Add ax, ay and az, with respect to a.

14. Add axy and $3xy$ with respect to xy.

15. Add $a(x+y)$ and $b(x+y)$.

16. From $a(x+2y+z)$ take $b(x+2y+z)$.

17. Add $6\times 2^3-7\times 2^3+2^3+3\times 2^3$.

V. SYMBOLS OF AGGREGATION

36. Removal of Symbols of Aggregation (§ 7, 3).

1. *Symbols of aggregation preceded by +, or no sign, can be removed, the signs of the terms remaining unchanged.*

Thus, $(a+b)+(c-d)$

is the same as $a+b+c-d$,

because the parentheses merely serve to group terms together, which grouping does not affect the *sum* total in any way.

2. *Symbols of aggregation preceded by − can be removed by changing the sign of every term within the symbols.*

Thus, $(6x+2y)-(3x-4y)-(-x+y)$

becomes $6x+2y-3x+4y+x-y$, or $4x+5y$,

since − before a parenthesis denotes that the following quantity is to be subtracted from what precedes, and this subtraction can always be effected by changing the signs of the subtrahend and adding (annexing) them to preceding quantity.

a. The + and − signs *before* a symbol *disappear* when the symbol is removed, since they denote an operation, and the operation is performed when the symbol is removed.

Care is needed in handling a vinculum quantity lest the student regard the sign immediately before the first term as belonging to this term instead of to the vinculum quantity. Thus, in $-\overline{2a+b}$, the − belongs to the whole quantity, $2a+b$, and + is understood before $2a$ inside the vinculum. In $-\overline{-2a+b}$ we see the sign − written before $2a$.

Remove symbols of aggregation and simplify, or add, in the following. After solving with pencil, solve mentally.

3. $(1-2x+3x^2)+(3-x^2+2x)$.
4. $(2x^2-3a^2x^2+9)-(x^2-5a^2x^2-3)$.
5. $(x^3-11xyz+3a)-(6xyz+7-2a-5xyz)$.
6. $-(5cz+4x^2y^3-8m)-(-cz+2x^2y^3-4cz)$.
7. $\overline{a-b-c}+\overline{b+c-d}-\overline{e-d-f}-\overline{-f+g-e}$.
8. $(3x^2+3y^2)-\{(x^2+2xy+y^2)-(-x^2+2xy+y^2)\}$.

a. We have here symbols of aggregation within other symbols of aggregation. One way for the beginner to remove such symbols is to remove only the innermost first, letting the other parts of the problem remain unchanged, then remove the next innermost, and so on.

However, all such symbols can be removed in one operation, beginning with the outermost, by noting the number of minus signs which affect any term, and making such term + if this number is even, and − if it is odd.

9. $a-\{b-(c-d)\}$. 10. $(a+2b)-[3b-(6a-6b)]$.
11. $7a-\{3a-[4a-(5a-2a)]\}$.
12. $1-\{2-(1-x+x^2)\}$.
13. $5a-[a+5x-\{a-x-\overline{3a-2x}\}]$.
14. $2x-[3y-\{4x-(5y-6x)\}]$.

37. Inserting Quantities within Symbols of Aggregation. From the preceding articles it follows, by reversing the operation, that

1. *Any terms can be inserted within a symbol of aggregation preceded by +, or no sign, without altering their signs.*

2. *Any terms can be inserted within a symbol of aggregation preceded by − by changing the sign before each term so inserted.*

3. Write $a^2+2ab+b^2-c^2+2cd-d^2$ as a binomial, placing the first three terms in one parenthesis and the last three in another, the latter preceded by $-$.

4. Write $12ax+12ay+4by-12az-15cx+2bz+6cy-3cz$ so that the terms which contain x may appear as one quantity, the terms which contain y as another, and the terms which contain z as another, the last subtracted from the sum of the other two.

5. A man who owns a mill worth $\$a$ has personal property worth $\$b$ and has $\$d$ in the bank. But he owes $\$e$ for his engine and owes his millwright $\$f$. However, the millwright is indebted to him for $\$g$ worth of flour. How can what he is worth be represented so as to keep the mill separate from the rest, the accounts separate from the other items, and the money due from the millwright separate?

Solution. $a+\{b+d-(e+\overline{f-g})\}$.

6. Remove the symbols from the answer to the preceding exercise.

VI. MULTIPLICATION

38. Multiplication, when the multiplier is a whole number, is the process of taking a quantity (the *multiplicand*) as many times as there are units in the *multiplier*. For other kinds of multiplication, as when the multiplier is a fraction, like $\frac{2}{3}$, or an irrational, like $\sqrt{5}$, this definition fails to have meaning. Such multiplications are extensions of the idea of multiplication by integral multipliers.

39. Laws assumed to hold in Multiplication.

I. Multiplication is Commutative.

Thus, $ab=ba$; $abc=acb=cba$.

a. Since $a\times b\times\frac{1}{c}\times d\times\frac{1}{e}$ must be commutative, by commutative law, then $a\times b\div c\times d\div e$, the same thing, is commutative.

Each letter, however, must not be separated from the operation symbol before it in changing the order of the operations.

Thus, $a\times b\div c\times d\div e=a\div c\times b\div e\times d=d\div c\times a\div e\times b=$ etc.

Because the operation symbols stand to the *left* of their quantities, *it is understood that the operations of multiplication and division when denoted by ×, :, ÷, ·, will be performed in any given expression in the order they come from left to right.* (See § 21, *a*).

For example, $12 \div 4 \times 3 \times 6 \div 2 = 27$; $12 \times 3 \div 4 \div 2 \times 6 = 27$.

II. MULTIPLICATION IS ASSOCIATIVE.

For example, $(ab)c = a(bc)$.

III. MULTIPLICATION IS DISTRIBUTIVE. *In other words, in every multiplication problem in both algebra and arithmetic, every term of the multiplicand must be multiplied by every term of the multiplier and the partial products added.*

It is well known to the pupil that this is always done in arithmetic in the multiplication of two such numbers as 2456 and 3985.

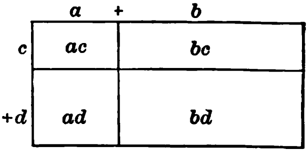
ARITHMETICAL EX.

$$\begin{array}{rl} 23 & \\ 25 & \\ \hline 115 & \\ 46 & \\ \hline 575 & \textit{Ans.} \end{array}$$

ALGEBRAICAL EX.

$$\begin{array}{rl} a+b & \\ c+d & \\ \hline ac+bc & \\ +ad+bd & \\ \hline ac+bc+ad+bd & \textit{Ans.} \end{array}$$

GEOMETRICAL ANALOGUE



40. The Signs in Multiplication.

1. When the multiplier is positive, the product has the same sign as the multiplicand. Thus, three credits of + $5 each give a credit of + $15, while four debts of − $7 each give a debt of − $28.

This suggests that when the multiplier is negative, the product should have the sign opposite to that of the multiplicand. We will now test this natural assumption to see if it is in agreement with our rules for addition and subtraction and the distributive law (§ 39) which we assumed to hold true for multiplication.

Thus,

$$\begin{array}{rcr} 9-4 & = & 5 \\ 7-3 & = & \times 4 \\ \hline 63-28 & & 20 \\ -27+12 & & \\ \hline 63-28-27+12 & = & 20 \end{array}$$

EXPLANATION. When 9 is multiplied by − 3, the product, by the rule just assumed, ought to have a sign opposite to that of + 9, and in multiplying − 4 by − 3, the product ought to have a sign opposite to that of −4, or be +12.

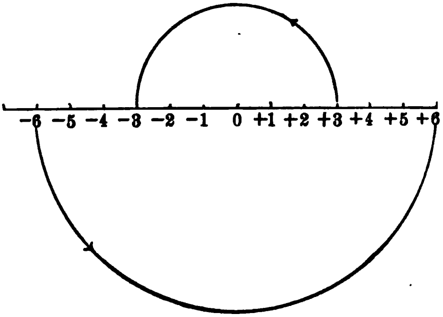
Testing our result, which we see by multiplying 5 and 4 together must equal $+20$, we find that, by our rule for addition,

$$63 - 28 - 27 + 12 = +20.$$

Hence we see that multiplication based on the distributive law and the two laws for signs just laid down agrees with addition. We have then this rule:

2. *In multiplication like signs in the two factors give a positive product and unlike signs give a negative product.*

3. How a positive multiplier gives the product the same sign as the multiplicand and a negative multiplier the opposite sign can be explained geometrically as follows: A positive multiplier simply repeats the number in the multiplicand. Hence, if the multiplicand is positive the product is positive, and if negative, the product is negative. But a negative multiplier *reverses the direction* of the multiplicand. Thus, $-1 \times +3 = -3$, that is, -1 as multiplier reverses the direction of $+3$, giving -3. Similarly in -2×-3, the multiplier -2 first *doubles* the multiplicand -3, and then reverses its direction, giving $+6$. Thus, a negative number $\times$ a negative number gives a positive product.



4. Considering a larger number of factors, we see readily that the sign of a product consisting of any number of positive factors and an *even* number of negative factors, as

$$a \times b \times c \times -d \times -e \times -f \times -g,$$

is positive, since by the associative law (§ 39), every pair of negative factors can be associated together, giving a single positive factor, so that all the factors become positive.

Thus, $\quad a \times b \times c \times (-d \times -e) \times (-f \times -g)$

becomes $\quad a \times b \times c \times de \times fg = +abcdefg.$

If a product consists, on the other hand, of any number of positive factors and an odd number of negative factors, the product is negative, since the last negative factor, which may be taken as coefficient (§ 39, I), will make the whole product negative.

Thus, $a \times b \times -c \times -d \times -e = -e \times a \times b \times (cd) = -abcde.$

Hence we have this rule for sign of product in multiplication:

5. *If there is an odd number of negative factors, the product is negative; otherwise it is positive.*

41. Exponents in Multiplication. The product of $2x^2$ and $3x^3$ is $6x^5$, since $2x^2 \times 3x^3 = 2xx \times 3xxx = 6x^5$, by the definition of an exponent, § 11, and by § 39, I. Similarly, $3x^4 \times 5x^6 = 15x^{10}$.

We see from these examples that the *coefficient* of the product is equal to the product of the coefficients of the factors, and the *exponents* of a letter in the factors are *added* to get the exponent of this letter in the product.

42. General Proof of the Exponent Law. Let a^m and a^n be the two factors of a product, both being integral powers of the same quantity, a.

Then, $a^m = a \times a \times a \times a \times a \cdots$ to m factors, by definition (§ 12),

and $a^n = a \times a \times a \times a \times a \cdots$ to n factors, by definition.

Hence,

$$a^m \times a^n = (a \times a \cdots \text{to } m \text{ factors})(a \times a \cdots \text{to } n \text{ factors})$$
$$= a \times a \times a \times a \times a \times a \times a \cdots \text{to } m+n \text{ factors.}$$

$\therefore\ a^m \times a^n = a^{m+n}$. (Since a^{m+n} means $a \times a \times a \cdots$ to $m+n$ factors.)

Therefore, we have proved generally that,

In multiplication the exponent of a factor in the product is equal to the sum of the exponents of this factor in the several quantities multiplied together.

43. Exercise in Multiplication. Perform the multiplication in the first 14 exercises mentally.

1. $+3 \times -4$¢. 2. -7×-8¢. 3. -6×19.

4. $3\,a^2b \times 2\,ab \times 5\,a^2bc$. *Ans.* $30\,a^5b^3c$.

5. $3\,x^2y \times -2\,xy$.

6. $4\,m^3n \times -7\,n^2$.

7. $(-2\,b + 3\,c) \times -8\,c$.

8. $4 \times 2^3 \times 3^2 \times -5^2 \times 2^2$.

9. $6 \cdot 5^2 \times 2 \times 5$.

10. $3^2 \cdot 5^2 \cdot 10 \times 2^2$.

11. Multiply $3\,y^2 + y - 1$ by $-24\,y^3$; $2 \times (-2\,x + 3\,y - 2\,z)$.

12. Multiply $5\,x + 4\,y$ by $3\,x - 2\,y$ (§ 39, III).
Ans. $15\,x^2 + 2\,xy - 8\,y^2$.

13. $(2\,ac^2 - 3\,by)(2\,c^3 - 3\,y^2)$; $(x^2 + y)(x + y^2)$.

14. $2\,a^m \times -3\,a^n$; $5\,x^m \times -2\,x^{2m}$; $2\,a^mb^n \times -2\,a^nb^m$.

15. $(5\,x^3 - 7\,x - 8)(x^2 - 2\,x + 1)$.

16. $(a^m + b^n + c^p)(a^m + 2\,b^n - 3\,c^p)$.

17. $(ax^2 + bx + c)(dx^2 - ex - f)$. Check with $a = 1$, $b = 2$, $c = 1$, $d = 3$, $e = 2$, $f = 5$, $x = 2$.

SOLUTION.

$$\begin{array}{llll}
ax^2 + bx & + c & & = 9 \\
\underline{dx^2 - ex} & \underline{- f} & & \underline{= \times 3} \\
adx^4 + bdx^3 + cdx^2 & & & 27 \\
\quad - aex^3 - bex^2 - cex & & & \\
\qquad\quad \underline{- afx^2 - bfx - cf} & & & \\
\multicolumn{4}{l}{adx^4 + (bd - ae)\,x^3 + (cd - be - af)x^2 - (ce + bf)x - cf = 27.}
\end{array}$$

18. $(mx^2 - 3\,nx + 4\,p)(px^2 - 5\,qx + 2\,r)$.

19. Show that
$$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x + (ab + ac + bc)x + abc.$$

20. $-3^2 \times 4$.

a. This problem may prove confusing to the pupil. Notice 3^2 as in $-a^2 \times 4$ has the coefficient -1 understood. Then $-1 \times 3^2 \times 4$ equals what by law of precedence of operations? If an algebraist desires to square -3 he writes it $(-3)^2$.

21. $6 \times (a \times b \times c)$.

22. $6 \times (a + b + c)$.

23. $4 \times (5 \times 3 \times 2)$.

24. $4 \times (5 + 3 + 2)$.

b. Evidently to multiply the *product* of several factors by a monomial multiplier, we must multiply only *one* of the factors by this multiplier, leaving the others unchanged; to multiply the *sum* of several terms by a monomial multiplier, we multiply each term by the multiplier, adding the products.

Perform the multiplications in the following exercises mentally without copying the problems:

25. $(x+3)(x+5)$.   26. $(2x+3)(3x+4)$.

27. If $3m-2$ feet is the width and $4m+3$ feet is the length of a rectangle, what is its area?

28. If $7m-3$ inches is the base and $2m+4$ inches is the altitude of a parallelogram, what is its area?

29. If $3y+7$ feet and $y+3$ feet are the parallel sides and $3y$ is the altitude of a trapezoid, what is its area?

30. At $2n-13$ cents apiece what will $5n-2$ oranges cost?

44. Powers and Roots of Monomials. Solve orally.

1. $(2a)^2$.   2. $(3m^3)^2$.   3. $(5ac)^2$.
4. $(-3a^2b)^2$.   5. $(\frac{1}{2}m^3)^2$.   6. $(22a^4)^2$.
7. $(-3a^mb^n)^2$.   8. $(a^2)^m$.   9. $(x^3)^3$.
10. $(-3x^3)^3$.   11. $(3a^n)^3$.   12. $(-6a^2b^n)^3$.
13. $(-2)^4$.   14. $(-2a^2)^5$.   15. $(a^m)^6$.
16. $(a^m)^n$.   17. $\sqrt{25a^2}=?$ *Ans.* $\pm 5a$. Why? Prove.
18. $\sqrt{16a^4}$.   19. $\sqrt{4a^{2m}}$.   20. $\sqrt[3]{-8a^3}$.
21. $\sqrt[3]{27m^9}$.   22. $\sqrt{64a^6b^4}$.   23. $\sqrt[n]{a^{2n}b^{3n}}$.

45. Arrangement of the Terms in a Polynomial and of the Factors in a Term. In the operations of multiplication, division, extraction of roots, etc., it is almost always very desirable to keep all quantities arranged in regular order, according to the ascending or descending powers of some *leading letter*, as

$$ax^3+bx^2-cx-d;\ 1+2m-m^3+3m^5.$$

The first of these expressions is arranged according to the *descending* powers of x, and the second according to the *ascending* powers of m.

In a single term (§ 14), it is the rule to arrange the letters *alphabetically*, as $5ab^3mx^2y$, though this is not essential.

The quantities heretofore given in addition, subtraction, and multiplication were quite regularly arranged according to these rules.

VII. DIVISION

46. Division is the inverse of multiplication, as subtraction is the inverse of addition. Thus division is the process of finding one factor when the product and the other factor are given.

Division bears much the same relation to subtraction that multiplication bears to addition. For integral numbers as multipliers, multiplication is often defined as a short method of addition when the numbers to be added are equal. Then division is the process of finding how many times the divisor can be subtracted from the dividend and what remainder will be left over.

47. Signs in Division. The rule for signs in division is quite like that for multiplication, viz.;

Like signs in the dividend and divisor give + for the sign of the quotient, and unlike signs give –.

This rule follows logically from the definition of division just given and the rule for signs in multiplication (§ 40, 2). Thus,

$+5 \times +6 = +30$, whence $+30 \div +5 = +6$;

or a + number ÷ a + number gives a + quotient.

$-5 \times -6 = +30$, whence $+30 \div -5 = -6$;

or a + number ÷ a – number gives a – quotient.

$+5 \times -6 = -30$, whence $-30 \div +5 = -6$;

or a – number ÷ a + number gives a – quotient.

$-5 \times +6 = -30$, whence $-30 \div -5 = +6$;

or a – number ÷ a – number gives a + quotient.

48. Exponents in Division.

The quotient of $10x^5 \div 2x^2 = 5x^3$, since $2x^2 \times 5x^3 = 10x^5$.

Similarly, $20x^7 \div 5x^3 = 4x^4$.

Thus, we see in these cases that the *coefficient* of the quotient equals the coefficient of the dividend divided by the coefficient of the divisor, and that the exponent of a letter in a quotient is equal to the exponent of the same letter in the dividend diminished by its exponent in the divisor.

To prove this generally, let a^p be the dividend and a^q the divisor, p and q being integral numbers.

Then $a^p \div a^q = a^{p-q}$, since $a^{p-q} \times a^q = a^p$ (by § 42). Hence

In division the exponent of a factor in the quotient equals the exponent of the same factor in the dividend diminished by its exponent in the divisor.

49. Exercise in Division. Perform the division in the first 12 exercises mentally.

1. $14a^2bc^3 \div 7ac$.
2. $-6a^4b^2 \div 2a^2b$.
3. $-24r^5s^2 \div -4rs$.
4. $-x^m \div x^2$.
5. $(a+b)^m \div (a+b)^n$.
6. $a^nb^2 \div ab$.
7. $14a^mc^p \div -2a^2c^3$.
8. $-16a^xy^b \div -8a^b$.
9. $(12 \cdot 3^5) \div (2 \cdot 3^2)$.
10. $(3xy + 2x^2y) \div xy$.
11. $6(a+x)^{y+2} \div 3(a+x)^{z-2}$.
12. $(6a^mb^4 + 3a^{m+1}b^3 - 4a^{m-2}b^2 - 12a^mb) \div 2ab$.
13. $(15x^2 + 16xy - 15y^2) \div (5x - 3y)$.

SOLUTION.

$$\begin{array}{r|l} 15x^2 + 16xy - 15y^2 & 5x - 3y \\ \underline{15x^2 - 9xy } & \overline{3x + 5y \ \textit{Ans.}} \\ 25xy - 15y^2 & \\ \underline{25xy - 15y^2} & \end{array}$$

14. Divide $6x^2 + 5xy - 4y^2$ by $3x + 4y$.
15. $(4x^5 + 4x - x^3) \div (3x + 2x^2 + 2)$.
16. $(m^3 - n^3) \div (m - n)$; $(a^5 - 243) \div (a - 3)$.
17. $(16x^3 - 29 + 27x - 46x^2) \div (8x - 3)$.

a. When any division ends with a remainder, test the work by multiplying divisor and quotient together and adding the remainder to the product.

18. $(21a^3 - 4a^2 - 12 - 42a) \div (4a + 2 - 3a^2)$.
19. Divide $1 + 2x$ by $1 - 3x$, getting 5 terms in the quotient. Prove answer.
20. $(x^{m+n} + x^ny^n + x^my^m + y^{m+n}) \div (x^n + y^m)$.

21. Find in the quickest way the numerical value of $x^4 + x^3 - 4x^2 + 5x - 3$ divided by $x^2 + 2x - 3$, when $x = 3$.

22. Find in the quickest way the numerical value of $(x^4 - 6x^3y + 9x^2y^2 - 4y^4) \div (x^2 - 3xy + 2y^2)$, when $x = 3$, $y = -1$.

23. Find in the quickest way $(x^4 - 6xy - 9x^2 - y^2) \div (x^2 + 3x + y)$, when $x = 4$, $y = 2$.

24. The area of a rectangle is $24a^2bc$ square inches and its base is $8bc$ inches. Find its altitude.

25. The cost of some articles was $36(m+n)(c-d)$ dollars and the price was $2(c-d)$ dollars. Find the number of articles.

26. The base of a triangle is $3abc$ feet and its area $24a^3bc^2$ square feet. Find its altitude.

50. Division by Detached Coefficients.

1. Divide $x^5 - 28x^3 + 8x^2 - 35x + 21$ by $x^3 - 5x^2 - 7$.

SOLUTION. We write the coefficients only in regular order, filling in missing powers of x with 0 coefficients.

$$\begin{array}{l|l}
1 + 0 - 28 + 8 - 35 + 21 & \underline{1 - 5 + 0 - 7} \\
\underline{1 - 5 + \ \ 0 - 7} & \underline{1 + 5 - 3} \\
\quad 5 - 28 + 15 - 35 & \text{or, } x^2 + 5x - 3 \quad \textit{Ans.} \\
\quad \underline{5 - 25 + \ \ 0 - 35} & \\
\qquad - 3 + 15 + \ \ 0 + 21 & \\
\qquad \underline{- 3 + 15 + \ \ 0 + 21} &
\end{array}$$

If the student does not understand this solution, if he will perform the division as in the preceding article, supplying always missing powers of x with 0 coefficients in every case, and will compare such solution with that just given, he will understand the process.

2. Divide $x^3 + 4x^2 + 7x + 6$ by $x^2 + 2x + 3$.

3. $(a^4 - 8a^2x^2 + 16x^4) \div (a^2 + 4ax + 4x^2)$.

SUGGESTION. Write the dividend $1 + 0 - 8 + 0 + 16$, thinking of it as standing for $a^4 + 0a^3x - 8a^2x^2 + 0ax^3 + 16x^4$.

4. $(15x^4 + 7x + 7x^3 + 15x^2 + 4) \div (3x^2 + 2x + 1)$.

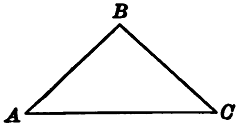
5. $(x^4 + x^2y^2 + y^4) \div (x^2 - xy + y^2)$.

6. $(1 + 5x^3 - 6x^4) \div (1 - x + 3x^2)$.

VIII. THEOREMS OF GEOMETRY

51. Theorems from Geometry. *For convenience of reference* we give here a *list* of the theorems of geometry which will be used in the remaining parts of the book. If a review of geometry is desired, the pupil should be asked to prepare anew the geometrical proofs of these propositions before he uses them in the solution of algebraical problems. If a review of geometry is not desired, the geometrical exercises themselves, of course, may be omitted.

1. *The angles opposite the equal sides of an isosceles triangle are equal, and, conversely, in any triangle, the sides opposite two equal angles are equal.*



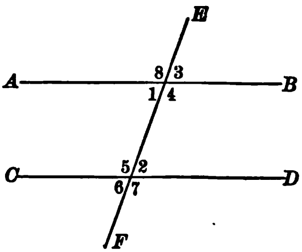
Thus, if $AB = BC$, then angle $C =$ angle A, and if angle $C =$ angle A, then $AB = BC$.

2. *An equilateral triangle is equiangular, and conversely.*

3. *If two straight lines intersect, the vertical angles formed are equal.*

4. *If two parallel lines are cut by a transversal, alternate interior angles are equal, and corresponding angles are equal, and conversely.*

Thus, if AB is parallel to CD, $\angle 1 = \angle 2$, $\angle 4 = \angle 5$, $\angle 3 = \angle 2$, etc.



5. *The opposite sides and angles of a parallelogram are equal, and the diagonals bisect each other.*

6. *The sum of the interior angles of a triangle equals two right angles.*

7. *The sum of the interior angles of an n-gon (n-sided polygon) equals $2n - 4$ right angles.*

8. *A radius perpendicular to a chord bisects it and also the subtended arc, and conversely.*

9. *Parallel chords intercept equal arcs on the circumference.*

10. *An inscribed angle, or an angle formed by a tangent and a chord, is measured by one half of the intercepted arc.*

11. *An angle formed by two chords intersecting within a circle is measured by one half of the sum of the intercepted arcs, and one formed by two secants meeting without a circle, or a secant and a tangent, or two tangents, is measured by one half of the difference of the intercepted arcs.*

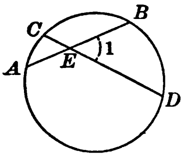


FIG. 1

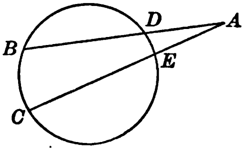


FIG. 2

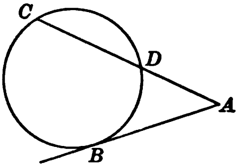
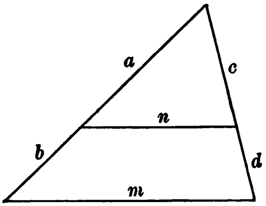


FIG. 3

Thus, $\angle 1$ (fig. 1) is measured by $\frac{1}{2}(AC + BD)$. Angle A (fig. 2) is measured by $\frac{1}{2}(BC - DE)$. Angle A (fig. 3) is measured by $\frac{1}{2}(BC - BD)$.

12. *If a line is drawn parallel to the base of a triangle, it divides the other two sides into segments which are proportional to each other and to the whole sides, and conversely.*



Thus, if n is parallel to m,

$$\frac{a}{b} = \frac{c}{d};\quad \frac{a}{c} = \frac{b}{d};\quad \frac{a+b}{b} = \frac{c+d}{d};$$

$$\frac{a+b}{a} = \frac{c+d}{c}.$$

13. *The bisector of an angle of a triangle, whether interior or exterior, divides the opposite side into segments which are proportional to the other two sides.*

Thus, if ABC is a triangle and CD and CD' are the bisectors of the interior and exterior angle C, we have

$$\frac{AD}{DB} = \frac{AC}{CB};\quad \frac{AD'}{D'B} = \frac{AC}{CB}.$$

14. *Two triangles are similar (that is, their corresponding angles are equal and their corresponding sides are proportional):*

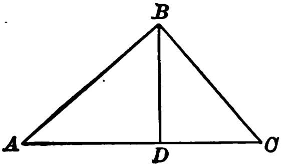
(*1*) *If they are mutually equiangular, or*

(*2*) *If an angle of one is equal to an angle of the other and the sides about these angles are proportional, or*

(*3*) *If their corresponding sides are proportional.*

15. *Two right triangles are similar if they have merely one acute angle of one triangle equal to an acute angle of the other.*

16. *If a perpendicular is dropped from the vertex of the right angle on the hypotenuse of a right-angled triangle:* (*1*) *the perpendicular is a mean proportional between the segments of the hypotenuse;* (*2*) *each leg is a mean proportional between the whole hypotenuse and its adjacent segment. Thus,*



$$\frac{AD}{DB}=\frac{DB}{DC}, \text{ and } \frac{AC}{AB}=\frac{AB}{AD}.$$

17. *If two chords intersect in a circle, their segments are reciprocally proportional. Or* (*fig. 1 under 11 above*),

$$\frac{AE}{CE}=\frac{DE}{BE}.$$

18. *If two secants intersect without a circle, they are reciprocally proportional to their external segments. Or* (*fig. 2 under 11*),

$$\frac{AB}{AC}=\frac{AE}{AD}.$$

19. *If a tangent and a secant intersect, the tangent is a mean proportional between the whole secant and its external segment. Thus* (*fig. 3, under 11*), $\frac{AC}{AB}=\frac{AB}{AD}$.

20. *Pythagorean Prop.* — *The square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides.*

21. *One side of a regular hexagon inscribed in a circle is equal to the radius.*

22. *One side of a regular decagon inscribed in a circle of radius r is x in the proportion* $\frac{r}{x}=\frac{x}{r-x}$.

23. *Theorems about the* **areas** *of a rectangle, parallelogram, triangle, trapezoid, circle, etc.*

IX. FORMULAS

52. **Formulas** are truths or rules expressed in algebraical symbols. They are largely used in pure and applied science.

1. If S = number of right angles in the sum of the interior angles of a polygon of n sides, by the theorem of the preceding article we have the formula, $S = 2n - 4$.

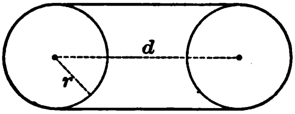
With this formula find the sum of the interior angles of polygons of 3, 4, 7, 11, 25 sides respectively. Change the result in each case to degrees.

2. If A = the number of right angles in one angle of an equiangular polygon of n sides, by the preceding exercise we have the formula, $A = \frac{2n - 4}{n}$.

Find from this formula one angle of equiangular polygons of 3, 4, 5, 6, 8, 10, 12, 30, 100 sides respectively. Change the result in each case to degrees.

3. If v is the number of degrees in the vertical angle of an isosceles triangle, make a formula giving the value of b, one of the base angles of the triangle, in degrees. Find b when $v = 25°$.

4. Show that the formula for the length l of a belt passing round two equal pulleys whose radii are r ft. and the distance between whose centers is d ft., is $l = 2\pi r + 2d$.



Find l when $r = \frac{1}{2}$ and $d = 3\frac{1}{2}$.

5. Show that the formula for the area a of a sector of a circle whose angle at the center has d degrees in it, r being its radius, is $a = \frac{d}{360}\pi r^2$.

Find a when $r = 7$ in. and $d = 40$.

6. Make a formula for the area S of a regular polygon of n sides one of whose sides is s and whose apothem is a. Find S when $n = 9$, $s = 1.5$, $a = 20.7$ in.

NOTE. Pupils should search for practical formulas and bring them to class, showing their use.

X. FUNCTIONS

53. A function of a quantity is any algebraical expression which contains this quantity, and which therefore *depends* for *its* value on the value of the quantity.

Thus, $\frac{3n+5}{2}$ is a function of n. If $n=2$, $\frac{3n+5}{2}=5\frac{1}{2}$; if $n=3$, the function equals 7; and so on for other values of n.

1. The length of a rectangle is x ft. and its width is $x-7$ ft. Find its area. *Ans.* x^2-7x sq. ft.

2. A is x years old now; how old will he be in $x+1$ years from now?

3. What number must be added to x to make y?

4. Two of the angles of a triangle have n and $2n+1$ degrees in them respectively. What function of n expresses the size of the third angle?

5. What must a be increased by to give $4-3a$?

6. The length, breadth, and thickness of a rectangular solid are $4y-3$ in., $3y-2$ in., and $y+4$ in. respectively. Find its solid contents.

7. By what must 5 be multiplied to make 3 less than n?

8. What is the cost in dollars of 40 electric lights at d dimes each?

9. How many hours will it take to walk m mi. at the rate of 200 ft. a minute?

10. What is the number of days needed by two men to reap a field if p men reap it in q days?

11. B was b years old when C was born. How old will B be when C is $2c$ years old?

12. Show that the sum of the interior angles of a polygon is the same function of n, whether obtained by joining the vertices to a point within the polygon or by drawing all the diagonals from the same vertex. Note in the latter case that the number

of triangles formed is 2 less than the number of sides of the polygon, and in the former case the angles about the point must be subtracted.

13. A man had \$$a$ invested at x%. What would be his total income throughout 3 years?

14. A man invested \$$y$ at 6% and the remainder of his total fortune of \$5000 at 5%. What was his total annual income?

15. A man sold a farm that cost him \$10,000 at the price of \$$x+5$ for 50 acres and \$$x$ for the remaining 60 acres, clearing money by the transaction. How much did he clear?

16. A wheelman rode 6 hr. at m mi. an hour. Then he decreased his speed and rode 4 hr. at $m-3$ mi. an hour. Lastly he rode 2 more hours at $m-6$ mi. an hour. What was his average speed?

17. In a certain family there are four children, each of which is h years older than the next younger. If y is the number of years in the age of the oldest, what expressions give the age of each of the others, and what is the sum of their ages?

XI. INTEGRAL EQUATIONS

54. An **equation** is a statement that two quantities have the same numerical value (§ 20).

An equation containing an unknown number is essentially an interrogative sentence. The sign "$=$" is equivalent to the verb "is," or to the phrases, "is the same number as," or "has the same numerical value as."

One of the glories of algebra is that by means of equations one solves with ease problems which students of arithmetic would probably find very puzzling or difficult.

55. The **solution of an equation**, or the finding of that value of its unknown number which makes the two sides of the equation have the same numerical value, is based on the matter already given in this chapter, and on the axioms.

56. An **axiom** is simply a truth assumed as the basis of a study.

57. The Axioms.

1. Addition Axiom: *Equals added to equals give equals.*

2. Subtraction Axiom: *Equals subtracted from equals leave equals.*

3. Multiplication Axiom: *Equals multiplied by the same number, or by equals, give equals.*

4. Division Axiom: *Equals divided by the same number, or by equals* (not zero), *give equals.*

5. Power Axiom: *Equals raised to the same power give equals.*

6. Root Axiom: *Corresponding roots of equals are equal.*

7. General Equational Axiom: *If the same operation be performed on two equal quantities, the results are equal.*

This last axiom holds true in general only for arithmetical numbers.

The axioms will be referred to as "Add. Ax.," "Sub. Ax.," "Mult. Ax.," etc.

8. Fundamental Reasoning Axiom: *Quantities equal to the same quantity are equal to each other.*

9. Axiom of Uniqueness: *The operations of addition, subtraction, multiplication, and division performed on quantities always give unique or "one-valued" results.*

58. Exercise in the Solution of Integral Equations.

1. Find value of x in equation $5x - 2 = 3x + 18$, and verify.

Solution. $5x - 2 = 3x + 18$. (Given equation.)

$5x - 3x = +2 + 18$. (By the subtraction axiom. Thus, the same quantity, $-2 + 3x$, is subtracted from each of the two equals, $5x - 2$ and $3x + 18$, giving the equal remainders, $5x - 3x$ and $2 + 18$.)

Or $2x = 20$. (By adding or simplifying on both sides of the equation.)

$x = 10$. (Division Axiom. The two equals $2x$ and 20 divided by the same number 2 give equals.)

Verification. $5 \times 10 - 2 = 3 \times 10 + 18$, or $48 = 48$.

Studying this solution, we see that when $3x$ disappeared from the right "member" of the equation and appeared in the left "member," its sign was changed from + to −. Similarly, when − 2 disappeared from the right member and appeared in the left member, its sign was changed from − to +. Evidently this rule will always hold true, since to "transpose" a quantity we may subtract it from each of the two members of the given equation. This causes it to disappear from the side where it now is, and to appear on the opposite side with its sign changed.

We have the following rule for solving an integral equation:

2. *Transpose all terms containing the unknown number to the left member of the equation and all known terms to the right member, changing the sign of every term transposed. Then add, or "collect," the quantities on each side, as in § 35. To finish the solution, divide both members of the resulting equation by the coefficient of the unknown. Last of all verify the answer found in the given equation.*

Solve and verify in the following, the first four mentally:

3. $2x-3=x+5$.
4. $5x-7=3x+15$.
5. $3p-25=p-9$.
6. $24n-49=19n-14$.
7. $7(m-3)=9(m+1)-38$.
8. $59(r-7)=61(9-r)-2$.
9. $5x+11=16-3x-4x$.
10. $6y-[7y-(8y-18)]=17$.
11. $(x+1)(x+2)=(x-3)(x-4)$.

a. One x^2 destroys the other in Ex. 11 when it is transposed. The word "cancel" is often used here, but it is desirable always to use the word "destroy" in such instances, reserving the word "cancel" for canceling the same factor in dividend and divisor or numerator and denominator.

12. $(x+1)(x+1)=[111-(1-x)]x-107$.

59. Problems involving the Solution of Integral Equations. The solution of a problem involves three steps: (1) Letting x or some initial letter represent the unknown number, and constructing the one or more functions described in the problem. (2) Changing a sentence in the problem which says two specified numbers are equal into an equation. (3) Solving the equation found and verifying the answer.

1. One base angle of an isosceles triangle is three times as great as the vertical angle. Find the number of degrees in each.

SOLUTION. Let x = the number of degrees in the vertical angle;

then $3x$ = the number of degrees in each base angle.

Now $x + 3x + 3x = 180$. (Sentence to equation.)

$7x = 180$. $\therefore x = 25\frac{5}{7}$.

a. The student may have been puzzled to see where the sentence was that was changed into the equation. The answer is it was implied in the statement of the problem. Strictly speaking, the problem should read: "One base angle of an isosceles triangle is three times as great as the vertical angle. Find the number of degrees in each, knowing that the sum of all three angles is 180°." It was also assumed that the number of degrees in the second base angle is the same as in the first base angle (§ 51, 1). In most problems the sentence is explicitly stated, but often it is implied, as in the preceding one.

2. Two angles are supplementary. One has $d - 1$ degrees and the other has $6(d + 1)$. How many degrees are there in each?

3. How many sides has an equiangular polygon four of whose angles equal seven right angles? See Ex. 2, § 52.

4. The area of the United States in 1850 was 559,452 sq. mi. more than 3 times what it was in 1800, and by 1910, exclusive of Alaska and the islands, it had increased 45,535 sq. mi. over what it was in 1850. Now the total increase from 1800 to 1910 was 2,260,675 sq. mi. What was the area of continental United States at each epoch?

5. The population of continental United States in 1850 was 383,408 less than 6 times what it was in 1790, and in 1910 it was 795,238 less than 4 times what it was in 1850. Now the increase from 1850 to 1910 was 10,992,404 more than 3 times what it was from 1790 to 1850. Find the population at each date and verify.

6. Of the six great continents Australia has 386,000 sq. mi. less area than Europe. Africa has 14,000 sq. mi. less than three times the area of Europe, and North America 2,081,000 sq. mi. less than Africa. South America has 828,000 sq. mi. less than twice the area of Europe, and Asia has 769,000 sq. mi. more than both Americas. The total area of all these continents is 52,153,000 sq. mi. Find the area of each continent.

XII. GRAPHS OF STATISTICAL DATA

60. Statistical Graphs. — If two quantities are related in such a way that as one changes in value the other also changes, this relative change can be pictured to the eye by means of a figure or diagram on squared paper.

Squared paper is not ordinary quadrille ruled paper, but specially made, the lines being ruled with great accuracy. The units of measure employed are commonly those of the metric system, the larger squares, ruled with heavier lines, being square centimeters, and the little squares having 2 millimeters on a side.

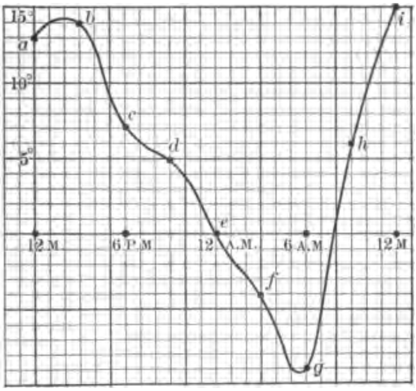
1. To show graphically the changes in temperature throughout the 24 hours of a day, having given the data in the table below.

SOLUTION. On a piece of squared paper we write 12 M. (noon) at any convenient centimeter intersection. At the right of 12 M. and underneath the sixth line marked with a dot we write 6 P.M.; then 12 A.M. underneath the 12th line, and so on to 12 M. again, underneath the 24th line. Passing upward from the first 12 M. point, opposite the fifth horizontal line we write 5°, opposite the tenth line, 10°, etc.

The table shows that at 12 M. the thermometer stood at 13°. To represent on the diagram this position of the mercury, a dot is placed at *a*, 13 units above 12 M.; then to represent the position of the mercury as 14° above 0 at 3 P.M., we start at 12 M., move 3 units to the right and then 14 units di-

TABLE

HOUR	READING	PT.
12 M.	13°	*a*
3 P.M.	14	*b*
6 P.M.	7	*c*
9 P.M.	5	*d*
12 A.M.	0	*e*
3 A.M.	– 4	*f*
6 A.M.	– 9	*g*
9 A.M.	6	*h*
12 M.	15	*i*



rectly upward, marking the point reached b. To show the position of the mercury at 6 P.M. as 7° above 0, we pass 7 units directly upward from the point marked 6 P.M. to the point c; and so on with the remaining data in the table. Through all the points a, b, c, d, e, f, g, h, i, a smooth curve is now drawn. This curve is the *graph* sought.

From the graph the approximate position of the mercury can be read off for any hour of the day. Thus, at 8 A.M. the mercury stood approximately at 1°.

2. Change the following data to tabular form and then construct the graph as in the preceding example: 12 M., 20°; 3 P.M., 15°; 6 P.M., 8°; 9 P.M., 3°; 12 A.M., – 2°; 3 A.M., – 5°; 6 A.M. – 10°; 9 A.M. + 2°; 12 M., 10°.

3. Construct a graph showing the changes of population of continental United States for every ten years from 1790 to 1910 from the following round number data:

1790 : 3,900,000
1800 : 5,300,000
1810 : 7,200,000
1820 : 9,600,000
1830 : 12,900,000
1840 : 17,100,000
1850 : 23,200,000
1860 : 31,400,000
1870 : 38,600,000
1880 : 50,200,000
1890 : 62,600,000
1900 : 76,100,000
1910 : 92,000,000

SUGGESTION. Let 1 cm. on the horizontal line represent 10 years, and taking the population for each year to the nearest hundred thousand, let one 2-millimeter unit denote one million on the vertical scale.

4. Find from the graph for Ex. 3 what the population was in 1875. In 1893.

5. The number of immigrants coming into the United States during the fiscal year ending June, 1908 was as follows: July 97,132; August 98,825; September 98,694; October 111,513; November 117,476; December 66,574; January 27,220; February 23,381; March 32,517; April 41,274; May 36,317; June 31,947. Construct graph from these data.

6. Construct other similar graphs, taking data directly from text-books or from one of the statistical almanacs. These almanacs can be had of news or book dealers at small cost, and are very valuable for reference. Among them may be named those issued by the *New York World* and *Chicago Daily News* newspapers.

61. Historical Notes. — Originally everything in algebraical solutions was written out in words. Later, abbreviations for these words were introduced. But it was not till the sixteenth century that symbols came into use to denote quantities, operations, and relations. Even at the present time, in the solution of many problems in arithmetic, words are freely brought into the statement of solutions. In the sixteenth century numerous writers on algebra introduced various signs. Some of these came into general use, while others dropped out. One writer, a Frenchman, Franciscus Vieta (François Viète), more than any other man of his time, made a quite general use of these symbols, and for this reason he has been called the father of algebra. Of course the substance of elementary algebra was then known, but the value of this possession was vastly increased by an appropriate notation.

Vieta* was born near La Rochelle, in France, in 1540, and died in 1603. He practiced law and was connected with both the parliament of Brittany and that of Paris. After 1580 he gave up most of his leisure to the study of mathematics and published numerous books and pamphlets. Being a man of some wealth, he disseminated his writings all over Europe at his own expense. This, no doubt, aided greatly in introducing a uniform symbolism. Vieta used consonants for known numbers and vowels for unknowns, and the vinculum as a symbol of aggregation (these being his personal contributions to our notation), employed + and − for addition and subtraction, but continued to use the letters *aeq.* (*aequalis*) for equals. He used Q (*quadratus*) and C (*cubus*) to denote the second and third powers respectively of the unknown, since exponents had not been thought of at that time.

Thus, for $2x^3 - 3x^2 + 4x = 12$ he would have written $2C - 3Q + 4N$ aeq. 12.

One day the ambassador from the Low Countries remarked to Henry IV, the king of France, that a countryman of his had propounded a problem to all mathematicians that had not thus far been solved, and asked if a Frenchman could be found to solve it.

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* See Frontispiece.

The king called for Vieta, and though the problem involved the solution of an equation of the 45th degree, Vieta secured two solutions in a few minutes, and gave them to the king written in pencil.

Henry was greatly impressed with the ability of Vieta and thought of him as one who might discover the Spanish cipher then in use by the Spanish government for the transmission of official dispatches. Though this cipher contained over 500 characters, and was changed from time to time, Vieta was soon able to read it, and the French used it greatly to their advantage in the war with Spain then raging. King Philip II complained to the Pope about the matter, charging that the French were using sorcery.

Vieta wrote on algebra, geometry, and trigonometry, but the service he rendered to algebra is his greatest glory.

Leonhard Euler was born in Basel, Switzerland, in 1707, and died in 1783. He studied at the University of Basel, and was a friend of the celebrated Bernouilli family. It was through the younger Bernouillis that he received a call when still a young man by the noted Queen Catherine of Russia to go to St. Petersburg, where he remained many years engaged in teaching and discovery. Later in his life Frederick the Great invited him to go to Berlin, where he continued his life work. Few men in history have been gifted with greater intellectual powers or with a more single devotion to learning.

Euler made a special study of physiology, of medicine, of botany, and of chemistry. He was learned in the classics and could repeat word for word the whole of Vergil's "Æneid," giving the first and last word of each page in the edition of the book he used. He is most noted, however, as a mathematician and astronomer. In mathematics he took many prizes from scientific societies, such as the French Academy.

When but 28 years of age, he solved a problem in three days which the most eminent mathematicians of the time asked for months to solve. The great exertion, however, brought on a fever, which resulted in the loss of the sight of one eye. This did not deter him from work, for only a year or so later he was winning prizes for solving the most difficult problems. In later life, soon

LEONHARD EULER (1707-1783)

after his return to St. Petersburg, a cataract grew over his remaining eye. The operation for its removal was not carefully enough done, and during the last years of his life Euler was nearly blind. It was at this time he was at work on calculations for explaining the inequalities or irregularities in the moon's motions, and these calculations and the data for them he had to carry in his head.

Late in life when almost blind he dictated to a tailor's apprentice, who knew nothing at all of mathematics, his "Introduction to Algebra," a book which is elementary in character, but which has been recognized as a standard text from that day to this. Euler's works if printed in their completeness would fill about 80 huge volumes. His researches aided greatly in the development of the various branches of mathematics, and he stands as one of the greatest mathematicians of all time.

His studies, carried on through a period extending considerably over half a century, much of the time in the face of the greatest obstacles, such as the difficulties of the subject and his own physical defect, set an example of whole-hearted love of knowledge which cannot fail to be an incentive to labor more diligently to all those who come to know of this man and his work.

CHAPTER II

THEOREMS IN MULTIPLICATION AND DIVISION, AND FACTORING

I. THEOREMS

62. A **theorem in algebra** is a fundamental truth expressed in *words*, whereas a **formula** is such a truth expressed in symbols. (See § 52.)

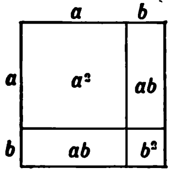
a. In the proofs that follow, and hereafter, we will use heavy-faced type to denote *any* algebraic quantities. Thus, $\mathbf{a}$ may stand for $2\mathbf{ab}$, or for $c+d$, and $\mathbf{b}$ may stand for $3c^2$, or $2m+3n-5p$, etc. The letters in ordinary italic type are understood to stand each for some number. Letters in heavy-faced type are to be understood as capable of denoting either a single letter or any combination of letters, figures, and signs, denoting a quantity. To be able to generalize in this way is one of the strong features of algebra.

63. Theorem I. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.*

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be any two quantities. Then by actual multiplication:

$$\begin{array}{l} \mathbf{a}+\mathbf{b} \\ \underline{\mathbf{a}+\mathbf{b}} \\ \mathbf{a}^2+\mathbf{ab} \\ \underline{\qquad+\mathbf{ab}+\mathbf{b}^2} \\ \mathbf{a}^2+2\mathbf{ab}+\mathbf{b}^2 \end{array}$$

Or, $(\mathbf{a}+\mathbf{b})^2=\mathbf{a}^2+2\mathbf{ab}+\mathbf{b}^2$.



Beside the multiplication we see a geometrical proof of this theorem. The square constructed on the sum of the lines $\mathbf{a}$ and $\mathbf{b}$ equals the square $\mathbf{a}^2$ constructed on $\mathbf{a}$, plus two rectangles $\mathbf{a}$ long and $\mathbf{b}$ wide, plus the square $\mathbf{b}^2$ constructed on $\mathbf{b}$.

The student should practice *pointing* to the different characters in the formula $(a+b)^2 = a^2 + 2ab + b^2$ which correspond to the words of the theorem as he repeats the theorem. *This course should be followed in the study of all the theorems.* Notice that in reading algebraical expressions very often we do not go regularly from left to right as in ordinary print.

Expand mentally the following by using the theorem:

1. $(2a+3b)^2$.

SOLUTION. $(2a+3b)^2 = (2a)^2 + 2(2a \times 3b) + (3b)^2$.
$= 4a^2 \quad + 12ab \qquad + 9b^2$. *Ans.*

2. $(5a+2b)^2$.
3. $(7a+5b)^2$.
4. $(3a^2+2y)^2$.
5. $(3x+4y)(3x+4y)$.
6. $(2m^3n+1)^2$.
7. $(a^3+2b^2)(a^3+2b^2)$.
8. $(\frac{1}{2}x+\frac{3}{5}y)^2$.
9. $(7+\frac{1}{2})^2$.
10. $(15+\frac{1}{3})(15+\frac{1}{3})$.
11. $(100+2)^2$.
12. $(25+3)^2$.
13. $(18+\frac{2}{3})^2$.
14. $(7\frac{1}{8})^2$.
15. $(16\frac{1}{4})^2$.
16. $(12\frac{5}{6})^2$.
17. $(9\frac{2}{5})^2$.
18. 24.5^2.
19. 8.2^2.
20. 25.5^2.
21. 100.5^2.
22. 80.4^2.
23. 20.8^2.

a. **The object in giving so many arithmetical problems here is to accustom the pupil to apply his algebra to arithmetic. Such calculations can often be performed mentally and much more quickly than by ordinary multiplication; or, they can be used to check the accuracy of ordinary multiplication.**

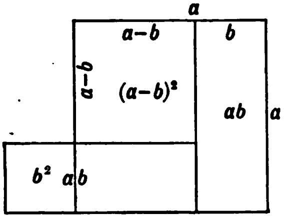
Factor the following quantities (this being the reverse operation to that we have been performing):

24. $9x^2+30xy+25y^2$.

SOLUTION. $\sqrt{9x^2} = 3x$; $\sqrt{25y^2} = 5y$. *Possible Ans.* $(3x+5y)^2$.
CHECK. $(3x+5y)^2 = (3x)^2 + 2(3x \times 5y) + (5y)^2 = 9x^2 + 30xy + 25y^2$.

25. $4m^2+12mn+9n^2$.
26. $x^2+10x+25$.
27. $a^2b^2+2abcd+c^2d^2$.
28. $36x^4+84x^2y^2+49y^4$.
29. $x^2+x+\frac{1}{4}$.
30. $a^{2n}+4a^n+4$.

64. Theorem II. *The square of the difference of two quantities equals the square of the first, minus twice the product of the first and second, plus the square of the second.*



Proof. Let a and b represent any two quantities. By actual multiplication, we have $(a-b)^2 = a^2 - 2ab + b^2$.

Proof by Geometry. $a^2 + b^2$ (or the area of the large square + the area of the small square below and to the left of the large square) diminished by the two ab rectangles equals the square on $a - b$.

Expand the following quantities by using Theorem II:

1. $(2m^2 - 3n)^2$.

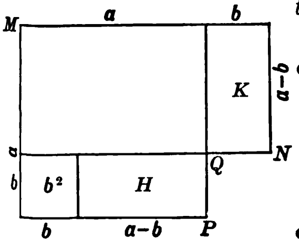
Solution. $(2m^2 - 3n)^2 = 4m^4 - 12m^2n + 9n^2$. *Ans.*

2. $(3mn - 5)^2$. 3. $(3abc - bcd)^2$. 4. $(100 - 3)^2$.
5. $(1 - \frac{2}{3}x^2)^2$. 6. $(16 - \frac{1}{4})^2$. 7. $(.4x^5 - .2)^2$.
8. $(12 - \frac{1}{8})^2$. 9. $(112 - \frac{1}{3})^2$. 10. $(87\frac{2}{3})^2$.
11. $(24\frac{4}{5})^2$. 12. $(17\frac{5}{8})^2$. 13. $(49\frac{5}{6})^2$.

Factor (reverse operation) the following:

14. $a^2 - 8a + 16$. (See § 63, Ex. 24.)
15. $m^4 - 20m^2 + 100$. 16. $4x^2y^2 - 20xyz + 25z^2$.
17. $36x^4 - 60x^2y + 25y^2$. 18. $1 - 6b^2c^3 + 9b^4c^6$.

65. Theorem III. *The product of the sum and the difference of two quantities equals the difference of their squares; and, conversely, the difference of the squares of two quantities equals the product of their sum and their difference.*



Proof. Let a and b represent any two quantities.

By actual multiplication

$$(a+b)(a-b) = a^2 - b^2.$$

Proof by Geometry.

$$H = K = (a-b)b.$$

Then, $MN = MQ + H = MP - b^2$.

or, $(a+b)(a-b) = a^2 - b^2$.

Expand the following quantities by using Theorem III:

1. $(2a + 3b)(2a - 3b)$.

SOLUTION. $(2a + 3b)(2a - 3b) = (2a)^2 - (3b)^2 = 4a^2 - 9b^2$. *Ans.*

2. $(3 + x)(3 - x)$.
3. $(m^6 - n^6)(m^6 + n^6)$.
4. $(3a^2 + 7b^3)(3a^2 - 7b^3)$.
5. $(a^m + b^n)(a^m - b^n)$.
6. 58×62, or $(60 - 2)(60 + 2)$.
7. $(17 + \frac{1}{3})(17 - \frac{1}{3})$.
8. 73×67.
9. $(26\frac{3}{4} \times 33\frac{1}{4})$.
10. $a^2 = (a + b)(a - b) + b^2$. (By transposing b^2.)

Thus, $67^2 = 70 \times 64 + 9 = 4489$.

11. Square in same way 27; 54; 89; 117.

Factor (reverse operation) the following:

12. $9a^2 - 16b^2$.
13. $25b^4 - 36$.
14. $4m^6 - 1$.
15. $49a^4 - \frac{1}{9}b^2$.
16. $a^{2m} - b^{2n}$.
17. $\frac{1}{4}x^2y^2z^2 - 25n^2$.

66. THEOREM IV. *The product of two binomials having a common term equals the square of the common term, and the algebraic sum of the other two terms times the common term, and the algebraic product of the other two terms.*

PROOF. Let x, a, and b represent any three quantities. Then

$$\begin{array}{l} x + a \\ x + b \\ \hline x^2 + ax \\ \quad + bx + a \\ \hline x^2 + (a + b)x + ab \end{array} \qquad \begin{array}{l} x - a \\ x - b \\ \hline x^2 - ax \\ \quad - bx + ab \\ \hline x^2 + (-a - b)x + ab \end{array} \qquad \begin{array}{l} x - a \\ x + b \\ \hline x^2 - ax \\ \quad + bx - ab \\ \hline x^2 + (-a + b)x - ab \end{array}$$

Expand the following quantities mentally by using Theorem IV:

1. $(x + 5)(x - 6)$.

SOLUTION. $(x + 5)(x - 6) = x^2 + (5 - 6)x - 30 = x^2 - x - 30$. *Ans.*

2. $(x + 2)(x + 3)$.
3. $(x - 4y)(x - 6y)$.
4. $(ax + 6)(ax - 9)$.
5. $(x^2 - 3xy)(x^2 + xy)$.
6. $(2a + 3)(2a + 7)$.
7. $(3ab - 7)(3ab - 5)$.
8. $(2m^2 + 5n)(2m^2 - 10n)$.
9. $(5c^n - 8d)(5c^n + 9d)$.
10. $(90 - 4)(90 + 3)$.
11. $(70 + 4)(70 + 5)$.

12. 64×67. **13.** 52×58. **14.** 47×54. **15.** 29×33.
16. 18×23. **17.** 97×105. **18.** 997×1004. **19.** 88×94.
20. 37×47. **21.** 16×29. **22.** 87×95. **23.** 63×69.

Factor (reverse operation) the following:

24. $x^2 + 11x - 26$.

SOLUTION. We find two numbers 13 and -2 whose *product* is -26 and *sum* is 11. $(x+13)(x-2)$. *Ans.*

25. $z^2 + 13yz + 12y^2$. **26.** $b^2 - 30b + 200$.
27. $a^{2n} - 18a^n + 45$. **28.** $4x^{2a} + 4x^a - 15$.
29. $y^8z^4 - 5y^4z^2n - 84n^2$. **30.** $y^2 + \frac{1}{2}y - \frac{3}{16}$.

67. THEOREM V. *The cube of the difference of two quantities equals the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.*

(In the cube of the *sum* of two quantities all the terms are positive.)

PROOF. By actual multiplication. Left to pupil.

1. $(2a^2 - 5b^3)^3$.

SOLUTION.
$$\begin{aligned}(2a^2 - 5b^3)^3 &= (2a^2)^3 - 3(2a^2)^2(5b^3) + 3(2a^2)(5b^3)^2 - (5b^3)^3\\ &= 8a^6 - 60a^4b^3 + 150a^2b^6 - 125b^9. \quad (\S\, 21,\ 1.)\end{aligned}$$

2. $(2a - 4b^2)^3$. 3. $(2a^2 - \frac{1}{2}b)^3$. 4. $(a^n - \frac{1}{4}b^2)^3$.

Factor (reverse operation) the following:

5. $27a^3 - 108a^2b + 144ab^2 - 64b^3$.

SOLUTION. $\sqrt[3]{27a^3} = 3a$; $\sqrt[3]{-64b^3} = -4b$. *Possible Ans.* $(3a - 4b)^3$.

CHECK.
$$\begin{aligned}(3a - 4b)^3 &= (3a)^3 - 3(3a)^2(4b) + 3(3a)(4b)^2 - (4b)^3.\\ &= 27a^3 - 108a^2b + 144ab^2 - 64b^3.\end{aligned}$$

6. $125x^3 - 75x^2 + 15x - 1$.

7. $8a^6 + 72a^4b^2 + 216a^2b^4 + 216b^6$.

68. THEOREM VI. *The square of the sum of any number of quantities equals the sum of their squares and the sum of twice the product of each quantity by each quantity that follows it.*

Proof. We will limit the proof to trinomials and quadrinomials, but the theorem holds generally. By actual multiplication

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc.$$

$$\begin{aligned}(a-b-c+d)^2 &= [a+(-b)+(-c)+d]^2\\ &= a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd.\end{aligned}$$

Expand the following by Theorem VI:

1. $(a-2b+5c)^2$.

Solution. $(a-2b+5c)^2 = a^2+4b^2+25c^2-4ab+10ac-20bc.$

2. $(3a-4m+2n)^2$.
3. $(2a^2-3bc+7b^2)^2$.
4. $(x-2y+z-5n)^2$.
5. $(a^m-2b^m-3c^m-2d^m)^2$.

69. Theorem VII. Divisibility Theorem.* *If a polynomial in* x *(§ 15) equals* 0 *when* $x=a$*, then the polynomial is exactly divisible by* $x-a$.

1. Show by the divisibility theorem that $x^2-7x+10$ is exactly divisible by $x-2$.

Solution. Putting $x=2$, $x^2-7x+10=4-14+10=0$.

Check. $(x^2-7x+10)\div(x-2)=x-5$.

2. Show that $x^2-27x-124$ is exactly divisible by $x+4$, and check.

Suggestion. Put $x=-4$. If polynomial equals 0 when this value of x is substituted, then by the theorem it is exactly divisible by $x-(-4)$, or by $x+4$.

3. By what is $2x^3+3x^2-4x-1$ exactly divisible?

Suggestion. Noting sum of coefficients, will $x=1$ answer?

---

* **Proof.** Let p = any polynomial in x, q = the quotient when p is divided by $x-a$, and r = the remainder, the division being continued until the remainder does not contain x.

Then, $p=q(x-a)+r$ (Since dividend = quotient × divisor + remainder.)

Now if $x=a$, $p=q(a-a)+r$. (By substituting a for x.)

Or, $0=q\times 0+r$. (Since $p=0$, by hypothesis, when $x=a$).

Hence, $r=0$,

or there is *no* remainder when p is divided by $x-a$.

4. By what is $x^3+6x^2+3x-10$ exactly divisible on inspection?

5. By what is $x^4+6x^3+3x^2+x+3$ exactly divisible? Put $x=+1$, -1, $+3$, and -3 in turn, these being the factors of the last term.

70. THEOREM VIII.* *The difference of any same powers of two quantities is exactly divisible by the difference of the quantities.*

For example, x^2-y^2, x^3-y^3, x^4-y^4, etc. are each exactly divisible by $x-y$.

71. THEOREM IX.† *The difference of the same even powers of two quantities is exactly divisible by both the difference and the sum of the quantities.*

Thus, x^2-y^2, x^4-y^4, x^6-y^6, etc. are each exactly divisible by both $x-y$ and $x+y$.

72. THEOREM X.‡ *The sum of the same odd powers of two quantities is exactly divisible by the sum of the quantities.*

Thus, x^3+y^3, x^5+y^5, x^7+y^7, etc. are exactly divisible by $x+y$.

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* PROOF. Let $x^n - a^n$ represent the difference of any same integral powers of two quantities, x and a. Now, if $x = a$,

$$x^n - a^n = a^n - a^n = 0,$$

and therefore, by § 69, $x^n - a^n$ is exactly divisible by $x - a$, no matter what the values of x and a are.

† PROOF. The difference of the same even powers is exactly divisible by the difference of the quantities by Theorem VIII. It remains to prove that the difference of the same even powers is also divisible by the sum of the quantities.

Let $x^n - a^n$ represent the difference of the same even powers of two quantities. Then, if $x = -a$, $x^n - a^n = (-a)^n - a^n = a^n - a^n = 0$,

because n is even; and hence, by the divisibility theorem, § 69, $x^n - a^n$ is exactly divisible by $x - (-a)$, or $x + a$.

‡ PROOF. Let $x^n + a^n$ represent the sum of the same odd powers of two quantities x and a. Then, if $x = -a$,

$$x^n + a^n = (-a)^n + a^n = -a^n + a^n = 0,$$

because n is odd; and hence by the divisibility theorem, § 69, $x^n + a^n$ is exactly divisible by $x - (-a)$, or $x + a$.

73. The Factoring of Binomials by Theorems VIII–X.* *Theorems VIII–X give* **one** *factor of any binomial which is the sum or difference of the same powers of two quantities. The other factor can be found by long division. Or, to find the remaining factor, divide the first term of the given binomial by the first term of the known factor, for the first term of the required factor. To find the next term of the required factor, divide its first term just found by the first term of the known factor and multiply the quotient by the second term of the known factor, ignoring signs. The third term is found from the second in the same manner as the second was found from the first; and so on to the last term.*

Whenever the known factor is a **difference**, *all the signs of the other factor are positive; when it is a* **sum**, *the signs of the second factor will be alternately positive and negative.*

The reasons for this rule can be easily seen by referring to actual long divisions.

Separate the following into their prime factors, proving the answers by actual multiplications if there is any uncertainty as to their correctness. Some or all should be performed mentally.

1. $m^3 - n^3$. 2. $8x^3 - y^3$. 3. $9x^2 - 16y^2$.
4. $x^5 - y^5$. 5. $27m^3 + 64n^3$. 6. $x^6 - y^6$.
7. $a^7 + b^7$. 8. $16c^4 - 81d^4$. 9. $a^6 + 125b^6$.
10. $a^9 + 8b^9$. 11. $27a^{12} + 8b^9$. 12. $a^{15} - \frac{1}{8}b^{18}$.

a. Theorems VIII–X cover the ground of the divisibility of binomials by *binomials.* Expressions which do not fall under one or other of these theorems are not divisible by binomial divisors. For example, $a^3 - b^3$, $a^5 - b^5$, etc. are not exactly divisible by $a + b$. Also $a^2 + b^2$, $a^4 + b^4$, $a^6 + b^6$, etc. are not exactly divisible by either $a + b$ or $a - b$. Test the truth of these statements by actual divisions, *and then state the results in the form of theorems.* Every student of algebra should know these truths.

b. It must be carefully noted, however, that it often happens that quantities which cannot be factored when tested in one way can be factored when tested in another.

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* The theorems just enunciated include the cases of the factoring of the difference of two cubes and the sum of two cubes. Thus,

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. (By Theorem VIII, and § 73.)

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$. (By Theorem X, and § 73.)

For example, while $a^6 + b^6$ is not divisible by either $a + b$ or $a - b$, it is divisible by $a^2 + b^2$ by Theorem X, since it can be written $(a^2)^3 + (b^2)^3$. Again, while such an expression as $a^4 + 4b^4$ is not divisible by any real *binomial* divisor, it will be shown later that it is divisible by a *trinomial* divisor.

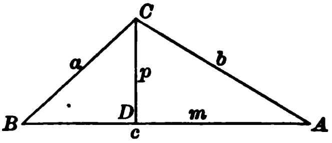
74. Promiscuous Exercise in the Use of the Theorems. Solve mentally.

Expand the first 23, and factor the remaining exercises. Quote the theorem used in each exercise.

1. $(2a + b)^2$.
2. $(a - 3b)^2$.
3. $(a + 2b)(a - 2b)$.
4. $(x - y)^3$.
5. $(x + 5)(x - 4)$.
6. $(a - b)(a - b)$.
7. $(3a^2 - 2b^2)^2$.
8. $(x - 7y)(x - 11y)$.
9. $(m^2 - 7n)(m^2 + 12n)$.
10. $(3a - 1)^3$.
11. $(2a^2 - 5b^2)(2a^2 + 5b^2)$.
12. $(2a - 11)(2a - 10)$.
13. $(3a^m - 2b^n)^2$.
14. $(2a^m + 7)(2a^m - 7)$.
15. $(x^3 - y^3)(x^3 + 21y^3)$.
16. $(2m - 3)^3$.
17. $(3x - 8)(3x + 6)$.
18. $(\frac{1}{2}m - \frac{1}{3}n)(\frac{1}{2}m + \frac{1}{3}n)$.
19. $(100 - 2)^2$.
20. 77×83.
21. $(2x - y - 3z)^2$.
22. $(3m^2 - 2n^3 + p^2)^2$.
23. $(a - 2b - 3c + 4)^2$.
24. $64x^3 - y^3$.
25. $9a^2 + 30ay + 25y^2$.
26. $a^5 - 32b^5$.
27. $x^2y^2 - 23xy + 60$.
28. $x^4 - 25y^4$.
29. $x^3 - 9x^2 + 27x - 27$.
30. $1 - (x - y)^2$.
31. $16a^2b^2c^2 + 24ab^2c^3 + 9b^2c^4$.
32. $a^3b^3 + 512$.
33. $a^2b^2 - 13abc + 40c^2$.
34. $a^9 - x^6$.
35. $a^4y^2 + 7a^2yz - 8z^2$.
36. $a^{2n} - b^{2n}$.
37. $m^{4p} - 3m^{2p} - 70$.
38. $8a^3 - 27b^3$.
39. $8x^3 - 60x^2y + 150xy^2 - 125y^3$.
40. $1 - (x - y)^3$.
41. $(a + b)^2 - 9c^2$.

75. Practical Applications of the Theorems of Multiplication.

1. *Prove that the square of the side opposite an acute angle in a triangle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other on that side.*



To prove $a^2 = b^2 + c^2 - 2\,cm$.

Proof. $a^2 = p^2 + \overline{BD}^2$ (§ 51.)

$= p^2 + (c - m)^2$

$= p^2 + c^2 - 2\,cm + m^2$.

(By Theorem II.)

$\therefore a^2 = b^2 + c^2 - 2\,cm$. Q.E.D. (Because $p^2 + m^2 = b^2$ by Pyth. Prop.)

2. Find the square of the side opposite an obtuse angle in a triangle.

3. Pythagoras stated that if $2\,a + 1$ and $2\,a(a + 1)$ are the legs of a right triangle, $2\,a^2 + 2\,a + 1$ is the hypotenuse. Prove that this is true by showing that the sum of the squares of the first two equals the square of the last.

4. Plato gave $4\,a$, $4\,a^2 - 1$, and $4\,a^2 + 1$ for the sides and hypotenuse, and Brahmagupta gave $2\,ab$, $a^2 - b^2$, and $a^2 + b^2$; verify their statements.

5. Put $a = 1, 2, \ldots$ in Ex. 3, 4, getting sets of integral numbers whose sides form right triangles.

6. Find how far a body will fall in the last second of its fall by calculating (simplifying) $\frac{1}{2}\,gt^2 - \frac{1}{2}\,g(t - 1)^2$.

II. FACTORING

76. Factoring is separating a quantity into other quantities whose product equals the given quantity. The quantities already factored and those to be factored in this chapter are rational integral quantities, or quantities that do not contain root expressions which cannot be found exactly.

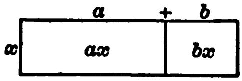
Factoring is an important topic. The difficulty of mastering the subject is greater than learning the fundamental operations because of the variety of processes and kinds of problems.

In the study that follows we will classify quantities into monomials, binomials, trinomials, quadrinomials, and multinomials. We will take up first a case which stands by itself, viz., that of monomial factors contained in polynomials.

77. Monomial Factors in Polynomials. A monomial is a factor of a polynomial if it is contained *in every term* of the polynomial.

Type forms: $ax + bx = (a + b)x$;

$3mx + 3my - 3mz = 3m(x + y - z)$.



GEOMETRICAL MEANING OF FACTORING. In the factored form $(a+b)x$, the area is expressed as the product of length by breadth; in the unfactored form, $ax + bx$, the area of each part is found and these parts are added. See p. 18.

1. Factor $6a^3 - 2am^2$ and prove answer by multiplication.

DIVISION SOLUTION. $2a\overline{)6a^3 - 2am^2}$
$\quad 3a^2 - m^2$

PROOF.
$$\begin{array}{r} 3a^2 - m^2 \\ 2a \\ \hline 6a^3 - 2am^2. \end{array}$$

DIRECT SOLUTION. $6a^3 - 2am^2 = 2a(3a^2 - m^2)$.

Factor the following, using both the division form and the direct solution form:

2. $4a^3 - 2ab$.
3. $20x^2y - 5y$.
4. $2m^3 - 6m^2n + 10mn$.
5. $a^2 - ab$.
6. $8a^3 - 12a^2 - 4$.
7. $ab - ac + ad - abc$.
8. $a(m + n) - 5b(m + n)$

SOLUTION. $m+n\underline{|\,a(m+n) - 5b(m+n)}$
$\qquad a \qquad -5b$

9. $2b(3x - y) + 3c(3x - y) + 2d(3x - y)$.

a. Throughout factoring the first step to be taken is to remove the monomial factors from the given quantities, if any are present. The other factor can then be separated into its factors. The presence of the monomial factor disguises many problems, and until it is removed, the student will not know how to proceed. This fact should not be forgotten.

1. Monomials

78. Factoring of Monomials. A monomial such as $3a^2b^3c$ is regarded as already factored. A quantity such as $48m^3n$ is factored when its coefficient is factored. Thus, $48m^3n = 2^4 \times 3m^3n$.

2. Binomials

79. Factoring of Binomials. Two different classes of problems may be distinguished, of which the first is altogether the more important.

I. *Binomials factored directly by one of the theorems: converse in Theorem III, and Theorems VIII, IX, X.*

See, for examples, §§ 65, 73.

II. *Binomials factored by inserting a quantity between the terms of the given binomial, and then subtracting the same quantity in a fourth term, thus changing the given binomial into a quadrinomial which is the difference of two squares.* See Ex. 13 below.

Factor the following by one of the theorems mentioned above:

1. $16x^4-9y^2$. **2.** x^3-27y^3. **3.** a^4-b^4.

4. $8m^6+125n^6$. **5.** $a^3b^3c^3-1$. **6.** z^5+1.

7. x^9+y^6. **8.** $a^{12}-16b^8$. **9.** $16x^4-625$.

10. $4x^6+108z^6$. **11.** $96m^4-486$. **12.** x^7+64xz^6.

13. Factor a^4+4b^4 and prove by multiplication.

Solution

(1) $(a^4+4a^2b^2+4b^4)-4a^2b^2$. (The term $4a^2b^2$, which is added and then subtracted, is twice the product of the square roots of a^4 and $4b^4$. Notice $4a^2b^2$ is chosen so that $a^4+4a^2b^2+4b^4$ is the square of a binomial as obtained by Theorem I, § 63.)

(2) $(a^2+2b^2)^2-(2ab)^2$. (By writing each part in the form of a square.)

(3) $(\overline{a^2+2b^2}+2ab)(\overline{a^2+2b^2}-2ab)$. (Factoring by converse of Theorem III, § 65.)

(4) $(a^2+2ab+2b^2)(a^2-2ab+2b^2)$. (Removing vincula and rearranging.)

(5) Prove by actual multiplication.

14. a^4+64b^4. **15.** x^4+324y^4. **16.** $4m^4+81p^4$.

3. Trinomials

80. The Factoring of Trinomials. Three different classes of problems may be distinguished, of which the first is much the most important.

I. *Trinomials factored by the trial method.* See Exs. 3, 4.

a. In certain special cases the trial method is materially aided by making use of Theorems I, II, IV. For instance, to factor a trinomial square, extract the square root of the first and last terms, and square the sum or difference of the roots by Theorems I and II to see if the result is the given trinomial. To factor a trinomial, the product of any two binomials having a common term, search for two quantities whose product is the third term of the given (arranged) trinomial, and whose algebraic sum times the square root of its square term equals the middle term of the given trinomial.

II. *Trinomial factored by changing the given trinomial into a quadrinomial which is the difference of two squares.* See Ex. 16.

III. *Trinomials factored by the divisibility theorem.* See Ex. 20.

1. Factor $9x^4+30x^2+25$, using Theorem I. See § 63, Ex. 24.

2. Factor $y^2-4yz-45z^2$, using Theorem IV. See § 66, Ex. 24.

3. Factor $15a^2-77am+10m^2$ by the trial method.

Solution.

$$\begin{array}{r} 3a-5m \\ 5a-2m \\ \hline 15a^2-25am \qquad\qquad \\ -6am+10m^2 \\ \hline 15a^2-31am+10m^2 \end{array} \qquad \begin{array}{r} 3a-2m \\ 5a-5m \\ \hline 15a^2-10am \qquad\qquad \\ -15am+10m^2 \\ \hline 15a^2-25am+10m^2 \end{array} \qquad \begin{array}{r} 15a-2m \\ a-5m \\ \hline 15a^2-2am \qquad\qquad \\ -75am+10m^2 \\ \hline 15a^2-77am+10m^2 \end{array}$$

Evidently this method consists in selecting coefficients for a pair of factors in such a way that the product of these factors will have the same coefficients for its first and last terms as those of the given trinomial. Then, by actual multiplication, the product is found and compared with the given trinomial. If the middle coefficients differ, another set of trial coefficients must be selected, and so on, till a set of coefficients and signs is found which gives the desired product.

b. If the given trinomial contains no monomial factor, neither can either of its factors. For, if one of two factors contains a factor, their product will contain this factor.

Thus, in the problem just solved it was useless to try the second set of factors $3a - 2m$ and $5a - 5m$, since $5a - 5m$ contains the factor 5 and the given trinomial does not.

We will give next another kind of trial method for factoring trinomials whose coefficients are large or give rise to many combinations of factors.

4. Factor $12x^2 - 41x - 15$.

Solution. $(12x)^2 - 41(12x) - 12 \times 15$. (Given quantity multiplied by the coefficient of x^2.)

This expression is now factored by the method of § 66. We seek for two numbers whose *product* is -180, or the product of the first and last coefficients of the given quantity, and whose *sum* is -41, or the coefficient of x in the given quantity. Starting with the divisor 2 and trying each divisor of -180 and its quotient, we look for a pair of factors whose sum is -41.

In this way we get 4 and -45. Then we write

$$(12x + 4)(12x - 45).$$

But we introduced the factor 12 above, and now must remove it from the factors found. Taking 4 out of the first factor and 3 out of the second, we have $(3x + 1)(4x - 15)$. *Ans.*

Proof. $(3x + 1)(4x - 15) = 12x^2 - 41x - 15$.

5. $x^4 + x^2 - 30.$

6. $y^2z^2 + 4yz + 4.$

7. $5a^4 - 10a^2b + 5b^2.$

8. $24x^2 - 29xy - 4y^2.$

9. $2c^2 - 13cd + 6d^2.$

10. $16a^{2n} - 24a^n + 9.$

11. $\frac{9}{16}y^2 - \frac{15}{2}y + 25.$

12. $x^{2m} - 19x^m + 90.$

13. $20 - 9x - 20x^2.$

14. $24a^2b^2c^2 - 37abc - 72.$

15. $25m^4 - 41m^2n^2 + 16n^4$. Factor each of the factors.

16. Factor $9x^4 + 38x^2y^2 + 49y^4$ by changing to a quadrinomial which is the difference of two squares.

Solution

(1) $(9x^4 + 42x^2y^2 + 49y^4) - 4x^2y^2$. (The term $42x^2y^2$ is found by taking twice the product of the square root of $9x^4$ and of $49y^4$. Then $4x^2y^2$ has to be subtracted in a fourth term to leave the given quantity unaltered.)

(2) $(3x^2 + 7y^2)^2 - (2xy)^2$. (Writing each part in form of square.)

(3) $[(3x^2 + 7y^2) + 2xy][(3x^2 + 7y^2) - 2xy]$. (Factoring by converse of Theorem III, § 65.)

(4) $[3x^2 + 2xy + 7y^2][3x^2 - 2xy + 7y^2]$. (Removing parentheses and rearranging.)

(5) Prove by actual multiplication.

17. Factor $9a^4 + 21a^2c^2 + 25c^4$ and prove.

18. Factor $a^4 + a^2b^2 + b^4$ and prove.

19. Factor $16a^4 - 17a^2b^2 + b^4$ and prove.

20. Factor $x^3 - 19x - 30$ by the divisibility theorem.

Solution. — We try the factors of 30.

If $x = \pm 1$, $x^3 - 19x - 30 = -48$, or -12.

If $x = \pm 2$, $x^3 - 19x - 30 = -60$, or 0; thus $x - (-2)$ is a factor.

If $x = \pm 3$, $x^3 - 19x - 30 = -60$, or 0; thus $x - (-3)$ is a factor.

If $x = \pm 5$, $x^3 - 19x - 30 = 0$, or -60; thus $x - 5$ is a factor.

Hence, $x^3 - 19x - 30 = (x + 2)(x + 3)(x - 5)$. Prove by actual multiplication.

21. $x^3 - 7x + 6$. 22. $y^3 - 13y + 12$. 23. $t^3 - 21t - 20$.

4. Quadrinomials

81. The Factoring of Quadrinomials. Four different classes of problems may be distinguished.

I. *Quadrinomials the product of two binomials, solved by inserting one pair of terms in one parenthesis, and the remaining pair in another parenthesis, and then, after removing monomial factors from each, dividing by a binomial factor now apparent.* See Ex. 1.

II. *Quadrinomials the difference of two squares, solved by placing three terms in parenthesis, and then factoring by converse of Theorem III.* See Exs. 8, 13.

III. *Quadrinomials the cube of a binomial, solved by extracting the cube root of the first and last terms, and checking the suggested answer by cubing the sum or difference of the roots by Theorem V.* See Ex. 18.

IV. *Quadrinomials factored by the divisibility theorem.* See Ex. 24.

1. Factor $3a^2x^3 - 4acx^2 - 6abx + 8bc$.

Solution. $(3a^2x^3 - 4acx^2) - (6abx - 8bc)$ (§ 37).

$$ax^2(3ax - 4c) - 2b(3ax - 4c)$$

$$\begin{array}{l|l} 3ax - 4c & ax^2(3ax-4c) - 2b(3ax - 4c) \\ \hline & ax^2 \qquad\qquad -2b. \end{array} \quad (\text{§ } 77, 8)$$

Proof. By multiplication.

Note. When the first term to be put in the second parenthesis is negative, the rule is to make the sign before the parenthesis negative. If one arrangement of the terms in parentheses fails to show a common binomial factor, try another. When the terms of the binomial in the second parenthesis have signs opposite to those in the first, change the sign before the second parenthesis, changing the signs inside at the same time.

Factor the following:

2. $ac - bd + bc - ad$.
3. $a^3 + a^2 + 2a + 2$.
4. $xy - 3y + 2x - 6$.
5. $cdx^2 - cxy + dxy - y^2$.
6. $15a^3 + 12a^2 - 10a - 8$.
7. $abcxy - b^2dy - acdx^2 + bd^2x$.
8. Factor $a^2 + 2ab - x^2 + b^2$, and prove.

Solution

(1) $(a^2 + 2ab + b^2) - x^2$. (Placing the three terms which form the square of a binomial in parenthesis.)

(2) $(a + b)^2 - x^2$. (Factoring the trinomial.)

(3) $(\overline{a + b} + x)(\overline{a + b} - x)$. (By converse of Theorem III.)

(4) $(a + b + x)(a + b - x)$. (Removing vincula.)

(5) **Proof.** By multiplication.

Factor the following:

9. $b^2 - 1 - 2ab + a^2$.
10. $4a^2 - 9b^2 + c^2 + 4ac$.
11. $x^2 - z^2 - 2xy + y^2$.
12. $(a - b)^2 - 4c^2$.
13. $4bc + a^2 - 4b^2 - c^2$.

Solution

(1) $a^2 - (4b^2 - 4bc + c^2)$. (The monomial square is here positive and must come first; the trinomial square must be inserted in a parenthesis preceded by minus sign.)

(2) $a^2 - (2b - c)^2$. (Factoring the trinomial.)

(3) $(a + \overline{2b - c})(a - \overline{2b - c})$. (By converse of Theorem III.)

(4) $(a + 2b - c)(a - 2b + c)$. (Removing vincula.)

(5) Prove by actual multiplication.

14. $4a^2 + 6bc - 9b^2 - c^2$.

15. $12bc + 4b^4 - 9b^2 - 4c^2$.

16. $24ab - 4a^2 + 9x^2 - 36b^2$.

17. $2ab - a^2 - b^2 + 1$.

18. $27c^2 - 12a^2 + 36ab - 27b^2$.

19. $3x^2y - 12xy^2 + 12y^3 - 3z^2y$.

20. $64x^6 - 144x^4y^2 + 108x^2y^4 - 27y^6$. (See § 67, Ex. 5.)

21. $8x^3 - 60x^2y + 150xy^2 - 125y^3$.

22. $216a^6b^3 + 108a^4b^2c + 18a^2bc^2 + c^3$.

23. $x^3 + 5x^2 - 18x + 8$. (See § 69.)

24. $12x^3 - 40x^2 + 13x + 30$.

Solution. A factor of the given quantity may be of the form $ax - b$, of which a is exactly contained in 12, and b is exactly contained in 30.

If $ax - b = 0$, $x - \frac{b}{a} = 0$ (Div. Ax.). If, when $\frac{b}{a}$ is substituted for x, the given polynomial in x reduces to 0, $\frac{b}{a}$ is a root. Thus, to find a factor of $12x^3 - 40x^2 + 13x + 30$, we will substitute for x fractions whose numerators are factors of 30, and whose denominators are factors of 12, as $\frac{5}{6}$, $\frac{5}{3}$, $\frac{5}{4}$, etc.

If $x = \pm 1$, $12x^3 - 40x^2 + 13x + 30 = 15$, or -35.

If $x = \pm 2$, $12x^3 - 40x^2 + 13x + 30 = -8$, or -252.

If $x = \pm \frac{3}{2}$, $12x^3 - 40x^2 + 13x + 30 = 0$, or -120.

Hence, $x - \frac{3}{2}$, or $2x - 3$, is a divisor of the given quantity. Dividing it by $2x - 3$, we get the quotient $6x^2 - 11x - 10$, whose factors are $2x - 5$ and $3x + 2$. Thus,

$12x^3 - 40x^2 + 13x + 30 = (2x - 3)(2x - 5)(3x + 2)$. *Ans.*

25. $12x^3 - 28x^2 + 17x - 3$.

Note. For convenience of reference we complete the subject of factoring with an article on multinomials. The factoring of such quantities is not often called for, even quadrinomials to be factored appearing much less often than binomials and trinomials. If thought desirable § 82 can be omitted, and referred to when needed. Only the last two exercises of § 85 are multinomials.

5. Multinomials*

82. The Factoring of Multinomials. A great number of kinds of problems might be given, but only six classes will be considered.

I. *Multinomials that are the difference of two squares, solved by converse of Theorem* III. See Ex. 1, below.

II. *Multinomials solved by placing sets of terms in parentheses, and factoring these parentheses quantities, thus making apparent a compound factor.* See Ex. 5.

III. *Multinomials separable into two trinomial factors solved by the trial method.* See Ex. 8.

IV. *Multinomials that are the square of a polynomial, solved by extracting the square root of the square terms, and squaring the sum of the roots by Theorem VI.* See Ex. 12.

V. *Multinomials that are the power of a binomial, solved by extracting the root of the first and last terms and raising the sum or difference of roots to power indicated using Newton's theorem,* § 166.

VI. *Multinomials solved by the divisibility theorem.*

Factor the following multinomials:

1. $a^2 + 2ab + b^2 - c^2 + 2cd - d^2$.

SOLUTION.
$$(a^2 + 2ab + b^2) - (c^2 - 2cd + d^2).$$
$$(a + b)^2 - (c - d)^2.$$
$$(\overline{a + b} + \overline{c - d})(\overline{a + b} - \overline{c - d}). \quad \text{(Theorem III.)}$$
$$(a + b + c - d)(a + b - c + d).$$

Prove by actual multiplication.

2. $4m^2 - 12mn + 9n^2 - p^2 - 4pq - 4q^2$.

3. $(a + b + c)^2 - (d + e + f)^2$.

4. $(a^2 + b^2 - c^2 - d^2)^2 - [2(ab + cd)]^2$.

Separate each of the *factors* of Ex. 4 into two factors.

5. $4x^2 - 16xy + 15y^2 - 4xz + 6yz$.

SUGGESTION. Put first three terms in a parenthesis and last two in another, and next factor each parenthesis quantity; then proceed as in § 77, S.

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* See note at end of p. 57.

6. $ax^2 + ax - 6a + x^2 + 7x + 12.$

7. $a^2(b - c) + b^2(c - a) + c^2(a - b).$

SUGGESTION. $a^2(b - c) - a(b^2 - c^2) + bc(b - c).$

8. $6x^2 - 19xy + 15y^2 + 7xz - 13yz - 20z^2.$

SOLUTION.

$$\begin{array}{rrrrrr}
2x - 3y + 5z \\
\underline{3x - 5y - 4z} \\
6x^2 & -\ 9xy & +\ 15xz & & & \\
 & -\ 10xy & & +\ 15y^2 & -\ 25yz & \\
 & & -\ 8xz & & +\ 12yz & -\ 20z^2. \\
\hline
6x^2 & -\ 19xy & +\ 7xz & +\ 15y^2 & -\ 13yz & -\ 20z^2.
\end{array}$$

EXPLANATION. The trial method is used. Coefficients are used which will give $6x^2$, $15y^2$, and $-20z^2$. Then, the two factors chosen are multiplied together. If the given polynomial results, the factors chosen are the ones sought; if it is not obtained, other combinations of coefficients and signs must be tried.

9. $6x^2 - 11xy + 3y^2 - 5xz + 11yz - 4z^2.$

10. $2a^2 + 6ax - 18a + 4x^2 - 24x + 36.$

11. $2a^2 - 4ab - 4ac + 2b^2 + 4bc + 2c^2.$

12. $a^2 + 9b^2 + 25c^2 - 6ab + 10ac - 30bc.$

SOLUTION. $\sqrt{a^2} = a$; $\sqrt{9b^2} = 3b$; $\sqrt{25c^2} = 5c.$
Then test $(a - 3b + 5c)^2$ by § 68.

13. $a^2 + 4b^2 + 9c^2 + 4ab - 6ac - 12bc.$

14. $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4.$

83. The Different Methods used in Factoring.

1. *Method of taking monomial out of polynomial.*
2. *By use of a theorem.*
3. *By trial.*
4. *Method which takes monomials out of parentheses sets.*
5. *Arranging as the difference of two squares and factoring.*
6. *Method for powers of binomials or squares of polynomials.*
7. *Method which uses divisibility theorem.*

III. OUTLINE FOR FACTORING

84. Outline for Factoring with Type Forms.

I. Monomials in Polynomials.

Take monomial factors out of all expressions *first.*

$$ab + ac = a(b + c);\quad ax + bx - cx = (a + b - c)x.$$

II. Binomials.

1. *Difference of two squares.*

$$a^2 - b^2 = (a + b)(a - b).$$

2. *Difference of same powers.*

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$
$$a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3).$$
$$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4).$$

. . . .

a. To get prime factors, factor first by Case 1, whenever it can be used to factor the given quantity or any of its factors.

3. *Sum of same odd powers.*

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$
$$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4).$$

. . . .

4. *Binomials changed to quadrinomials,* the difference of two squares.

$$\begin{aligned} a^4 + 4b^4 &= (a^4 + 4a^2b^2 + 4b^4) - 4a^2b^2. \\ &= (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2). \end{aligned}$$

III. Trinomials.

1. *Trinomial squares.*

$$a^2 + 2ab + b = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$

2. *Trinomials factored by trial method.*

3. *By changing trinomial to quadrinomial.*

$$a^4+a^2b^2+b^4=(a^4+2\,a^2b^2+b^4)-a^2b^2.$$
$$=(a^2+ab+b^2)(a^2-ab+b^2).$$

4. *By divisibility theorem.*

IV. QUADRINOMIALS.

1. *The product of two binomials.*

$$ab+ac+bd+cd=a(b+c)+d(b+c)$$
$$=(a+d)(b+c).$$

2. *The difference of two squares.*

$$a^2+2\,ab+b^2-c^2=(\overline{a+b}+c)(\overline{a+b}-c)$$
$$a^2-b^2+2\,bc-c^2=a^2-(b^2-2\,bc+c^2)$$
$$=(a+\overline{b-c})(a-\overline{b-c}).$$

3. *The cube of a binomial.*

$$a^3-3\,a^2b+3\,ab^2-b^3=(a-b)^3.$$

4. *Divisibility theorem.*

V. MULTINOMIALS.

1. *Problems solved like 1 in quadrinomials.*

2. *Difference of two squares like 2 in quadrinomials.*

3. *Solved by trial method like 2 in trinomials.*

4. *Square of a polynomial.*

$$a^2+b^2+c^2+2\,ab+2\,ac+2\,bc=(a+b+c)^2.$$

. . . .

5. *Powers of binomials.*

$$a^4-4\,a^3b+6\,a^2b^2-4\,ab^3+b^4=(a-b)^4.$$

. . . .

6. *By divisibility theorem.*

85. Promiscuous Exercise in Factoring. — In factoring the following exercises, *first set aside monomial factors*, if any are present. Then factor by one of the methods of either of the two preceding articles. If difficulty is experienced, note under which class, whether a binomial, trinomial, etc., the given quantity belongs, and then look for a corresponding type of problem under the corresponding class, and follow the model solution for this type.

Separate each quantity into its *prime* factors, and prove the answers right by a theorem of multiplication or by actual multiplication, if any doubt exists as to the correctness of the result.

1. $a^2 - 25\,b^2$.
2. $a^2 - 12\,a + 36$.
3. $20\,a^3x^3 - 45\,axy^2$.
4. $3\,x^2 + 8\,x + 4$.
5. $27\,n^3 - 8\,p^3$.
6. $x^3 + x^2 + x + 1$.
7. $H^2 - 3\,H + 2$.
8. $x^4 - 13\,x^2 + 36$.
9. $4\,t^2 + t - 14$.
10. $6\,x^3 - 7\,ax^2 - 20\,a^2x$.
11. $x^4 - 7\,x^2 - 18$.
12. $m^6 + 7\,m^3 - 8$.
13. $1 - a^4$.
14. $mnpq + 2 + pq + 2\,mn$.
15. $\pi R^2 - \pi r^2$.
16. $x^2 + (a + b)x + ab$.
17. $x^{16} - y^{16}$.
18. $8\,x^3 - 24\,x^2 + 25$.
19. $8\,a^2 - 2\,a - 3$.
20. $2\,x^4 + x^2 - 28$.
21. $8\,c^2 - 6\,cd - 5\,d^2$.
22. $18\,x^2 + 33\,axy + 14\,a^2y^2$.
23. $\frac{1}{2}\,ab + \frac{1}{2}\,ac$.
24. $\frac{1}{2}\,ab + \frac{1}{2}\,ab' + \frac{1}{2}\,ab''$.
25. $\frac{4}{3}\,\pi R^3 - \frac{4}{3}\,\pi r^3$.
26. $bk - ck + dk + ek$.
27. $20 - 9\,x - 20\,x^2$.
28. $m^3x + m^3y - n^3x - n^3y$.
29. $x^4 + x^2 + 1$.
30. $my^4 + 16\,mx^4 - 12\,mx^2y^2$.
31. $2\,am^5 + 2\,an^5$.
32. $x^9 - 64\,x^3 - x^6 + 64$.
33. $(a + b)^2 - c^2$.
34. $(x - y)^2 - 14(x - y) + 40$.
35. $\frac{1}{6}\,\pi d^3 + \frac{1}{6}\,\pi D^3$.
36. $2\,\pi r^2 + 2\,\pi rh$.
37. $\frac{wty}{2} - \frac{wtr}{6}$.
38. $a(a - c) - b(b - c)$.
39. $\frac{2\,\pi rh}{3} + \frac{2\,\pi r^2}{4}$.
40. $(x^2 + y^2 - z^2)^2 - 4\,x^2y^2$.

41. $125a + 64ax^{3n}$.
42. $2ab - a^2 - b^2 + 1$.
43. $x^{2m} - (y - z)^{2m}$.
44. $21x^3 + 26x^2 + 25$.
45. $2ma^{6m} - 2mb^{6m}$.
46. $x^3 - 6x^2 + 6x - 1$.
47. $9x^3 + 5x + 50$.
48. $1 + (b - a^2)x^2 - abx^3$.
49. $a^2b^2 - a^2 - b^2 + 1$.
50. $1 - (x - y)^3$.
51. $x^4 - 7x^2y^2 + y^4$.
52. $x^3 + y^3 + 3xy(x + y)$.
53. $a^2y^3 - b^2x^2y - a^2dy^2 + b^2dx^2$.
54. $x^3 + x^2 - 22x - 40$.
55. $6x^3 - 3x^2 - 33x - 6$.
56. $a^{5n} - b^{5n}$.
57. $A^2 - B^2 + C^2 - 2AC$.
58. $6m^4 - 384n^4$.
59. $8a^3 + 6a^2y^2 - 9ay^4 - 27y^6$.
60. $125x^{15} - 300x^{14} + 240x^{13} - 64x^{12}$.
61. $4a^2b^2 - 169c^2 + 6abd + 39cd$.
62. $4A^2 - 9B^2C^2 + 30BCD - 25D^2$.
63. $2a^2 + 6ax - 18a + 4x^2 - 24x + 36$.
64. $2x^2 - xy - 3y^2 - 5yz - 2z^2$.

86. Prime Quantities. All the quantities given in the preceding articles were factorable, but quantities *taken at random* are more likely to be prime than composite.

Thus, $3x^2 + 7x + 9$, $3x^2 + 7xy - 9xz + z^2$, $9x^2 - 11y^2$, etc., are all prime quantities in terms of rationals. We have seen that the sum of two squares is almost always prime. The expressions $x^2 + xy + y^2$, $x^2 - xy + y^2$, are prime.

It must be remembered, however, as already stated, if fractional exponents are allowed, some of these prime quantities may be factored. See Chapter VII on Fractional Exponent Quantities.

IV. APPLICATIONS OF FACTORING

87. Applications of Factoring. Factoring can be used in the solution of quadratic equations, in arithmetic, and in geometry. It is employed more or less throughout the study of algebra.

88. Application of Factoring in Arithmetic.

1. If a is the amount, b the base, and r the rate in percentage, $a = p + pr$. Factor the right member and state the result in words.

2. To make a rule for squaring any number increased by $\frac{1}{2}$, let $a =$ the number.

Then, $(a+\frac{1}{2})^2 = a^2 + a + \frac{1}{4}$. (By Theorem I.)

$= a(a+1) + \frac{1}{4}$. (By factoring a out of first two terms.)

Hence to square any number plus $\frac{1}{2}$, *multiply the number by one more than itself and add* $\frac{1}{4}$.

Square by this rule the following: $6\frac{1}{2}$, $11\frac{1}{2}$, $25\frac{1}{2}$, 100.5, 7.5.

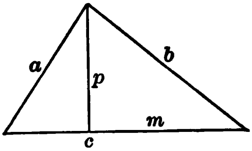
3. Calculate $2 \cdot 3^2 \cdot 17 + 2 \cdot 3^2 \cdot 13 - 2 \cdot 3^2 \cdot 9$.

SUGGESTION. Take $2 \cdot 3^2$ out first.

4. Calculate $2^2 \cdot 5^3 \cdot 7^3 + 2 \cdot 5^4 \cdot 7^3 - 2^2 \cdot 5^4 \cdot 7^2$.

89. Factoring applied in Geometry.

1. Derive the formula for the area of a triangle when the three sides are given.



Given the triangle whose sides are a, b, c, to find its area.

Referring to result under Ex. 1, § 75, we have

$$a^2 = b^2 + c^2 - 2\,cm,$$

whence $m = \dfrac{b^2 + c^2 - a^2}{2\,c}$. (By transposing both a^2 and $2\,cm$, and then dividing both sides of equation by $2\,c$.)

To find the area of the triangle, we reason now as follows: The area of the triangle is evidently $\frac{1}{2}\,cp$. We know c, and we can find p by using the value of m just found as one side of the right triangle whose three sides are m, p, and b. Thus,

$$p^2 = b^2 - m^2 = b^2 - \left(\frac{b^2 + c^2 - a^2}{2\,c}\right)^2.$$ (By substituting its value for m.)

$$p^2 = \left(b + \frac{b^2 + c^2 - a^2}{2\,c}\right)\left(b - \frac{b^2 + c^2 - a^2}{2\,c}\right).$$ (Converse Th. III.)

$$p^2 = \left(\frac{2\,bc + b^2 + c^2 - a^2}{2\,c}\right)\left(\frac{2\,bc - b^2 - c^2 + a^2}{2\,c}\right).$$ (Reducing mixed quantities to improper fractions.)

$$p^2 = \frac{(b^2+2bc+c^2)-a^2}{2c} \times \frac{a^2-(b^2-2bc+c^2)}{2c}.$$ (Arranging terms.)

$$\frac{c^2p^2}{4} = \frac{(b+c+a)}{2}\,\frac{(b+c-a)}{2} \times \frac{(a-b+c)}{2}\,\frac{(a+b-c)}{2}.$$

(§ 81, Exs. 8, 13 and multiplying both sides by $\frac{c^2}{4}$.)

Now let $s = \frac{a+b+c}{2}$; then $s-a = \frac{b+c-a}{2}$; (By Sub. Ax.)

$s-b = \frac{a-b+c}{2}$; $s-c = \frac{a+b-c}{2}$.

Then, $\frac{c^2p^2}{4} = s(s-a)(s-b)(s-c)$. (By substituting in the preceding expression.)

$\therefore \frac{cp}{2} = \sqrt{s(s-a)(s-b)(s-c)}$. (By extracting the square root of the two equals, Root Axiom.)

But $\frac{cp}{2}$ is the area of the triangle sought. Therefore the right member is equal to the area of the triangle. Notice that we know s when the three sides are given, it being their half sum. The letter s is taken to denote the *half* sum instead of the sum itself, because the formula just found comes out without fractions by so doing. This formula for the area of a triangle is one of the most symmetrical and beautiful in the whole range of mathematics. Its derivation, as we have just seen, is largely a matter of factoring. Exercise in the use of this formula will be given later.

90. Application of Factoring to the Solution of Quadratic Equations. The **degree** of an integral equation equals the exponent of the highest power of its unknown quantity. **Simple** equations are those of the first degree, and **quadratic equations** are of the second degree (§ 18).

1. Solve the equation $x^2 - 4x = 32$ and verify.

SOLUTION

$x^2 - 4x - 32 = 0$. (Right member made 0 by transposing 32, by Sub. Ax.)

$(x-8)(x+4) = 0$. (Factoring left member.)

$x - 8 = 0$, $x + 4 = 0$. (Setting *each* factor equal to zero.)

$\therefore x = 8$, *Ans.* $x = -4$, *Ans.* (Transposing known terms to right members, Sub. Ax.)

VERIFICATION. $8^2 - 4 \times 8 \equiv 32$. $(-4)^2 - 4(-4) \equiv 32$.

Solve the following and verify both roots:

2. $x^2 + 6x = 27$.

3. $3x^2 + 2x - 85 = 0$.

4. $5x^2 - 10x = x^2 - 170x - 700$.

5. $6x^2 + 4x = 3x^2 + 12x + 3$.

6. Find three consecutive numbers such that the sum of their squares shall be 434.

7. The perimeter of a rectangular lot is 90 ft. and its area 450 sq. ft. Find its dimensions.

8. A rectangular garden is 12 rods longer than wide and contains 1 acre. Find its dimensions.

91. Solution of Quadratics by Completing the Square.

1. Solve $x^2 + 6x - 50 = 0$, by completing the square.

SOLUTION

$x^2 + 6x = 50$. (Sub. Ax., known quantity to right member.)
$6x \div 2\sqrt{x^2} = 3$; $3^2 = 9$. (See explanation below.)
$x^2 + 6x + 9 = 59$. (Add. Ax., adding 9 to both sides.)
$x + 3 = \sqrt{59}$. (Root Ax., extracting square roots of equals.)
$x = \sqrt{59} - 3$. (Sub. Ax.)

EXPLANATION. This problem cannot be solved by the factoring method. Hence we turn to the method by completing the square.

By transposing the known term to the right member, we have remaining on the left side a binomial. This cannot be a square, since the square of a monomial is a monomial, and the square of a binomial is a trinomial. We must change the left member into a trinomial square by adding the proper term.

Let us consider that our problem is like having $a^2 + 2ab$ of $a^2 + 2ab + b^2$ given, to find b^2. To get b^2 we divide the second term $2ab$ by twice the square root of a^2, and square the quotient.

2. $x^2 + 12x - 40 = 0$.   3. $4x^2 - 12x = 39$.

a. By means of the completing the square process prime quantities can be separated into factors containing radicals, as will be made clear in a later chapter.

CHAPTER III

HIGHEST COMMON FACTOR — LOWEST COMMON MULTIPLE — FRACTIONS

I. HIGHEST COMMON FACTOR

92. A **common factor** of two or more quantities is a factor that appears in each of them. The **highest common factor** (h. c. f.) of two or more quantities is the product of all the prime factors common to each.

93. Finding Highest Common Factors by Factoring.

1. Find the highest common factor of $6\,a^2b$, $9\,a^3b$, and $24\,a^4b^3$.

SOLUTION. By inspection we see that $3\,a^2b$ is the h. c. f., since any quantity with a greater coefficient or higher exponents would not be contained in each, as $6\,a^3b^2$. The quantities ab and $3\,ab$, though common factors, are not the highest, a^2 being higher than a.

2. $6\,xy^2,\ 8\,x^2y,\ 12\,x^2y^2.$

3. $9\,m^3n,\ 12mn^3,\ 15\,m^3n^4.$

4. $7\,a^2b^2,\ 21\,a^3b^3,\ 14\,a^2b^4.$

5. $8\,a^mb^n,\ 12\,a^{2m}b^{3n},\ 15\,a^mb^{4n}.$

6. $x^2-y^2,\ x^3-y^3,\ x^2-7\,xy+6\,y^2.$

SOLUTION.
$$\begin{aligned} x^2-y^2 &= (x+y)(x-y).\\ x^3-y^3 &= (x-y)(x^2+xy+y^2).\\ \underline{x^2-7\,xy+6\,y^2} &\underline{= (x-6\,y)(x-y)}.\\ \text{h. c. f.} &= x-y. \quad \textit{Ans.} \end{aligned}$$

7. $x^2-2\,x-3,\ x^2+x-12,\ x^2-x-6.$

8. $2\,a^3-2\,ab^2,\ 4\,b(a+b)^2.$

9. $c^2x^2-d^2,\ acx^2-bcx+adx-bd.$

10. $2\,x^2-9\,x+4,\ 2\,x^2-11\,x+5,\ 2\,x^2+3\,x-2.$

11. $8\,a^3+1,\ 16\,a^4+4\,a^2+1.$

***94. Finding Highest Common Factors by Continued Division.** To explain this process two simple principles are needed:

A. *A divisor of a quantity is a divisor of any number of times that quantity.*

Thus, if 6 is contained in 18, it is contained in any integral number of times 18; if $3a$ is contained in $9a^2$ it is contained in $3b \times 9a^2$, or $27a^2b$.

B. *A divisor of each of two quantities is a divisor of their sum or difference.*

Thus, if 6 is contained in 66 and in 42 it is contained in their sum, 108, or in their difference, 24. It is evidently contained in their sum or difference the sum or difference of the numbers of times it is contained in each.

1. Find the greatest common divisor of 258 and 731 by continued division, and justify the process.

SOLUTION

```
258)731(2
    516
    215)258(1
        215
         43)215(5
            215
```

EXPLANATION OF PROCESS. 731 is divided by 258, which is contained twice with a remainder, 215. Then the *divisor*, 258, is divided by the *remainder*, and so on, the divisor each time becoming the new dividend, and the remainder becoming the divisor. The first divisor which is exactly contained in its dividend is the h. c. f. sought.

JUSTIFICATION OF PROCESS. (1) Is any divisor of 258 a divisor of 516? By which principle? Is any common divisor of 731 and 516 a divisor of 215? By which principle? Does it follow then that any common divisor of 258 and 731 must be contained in 215?

(2) Again, is any common divisor of 215 and 516 a divisor of 731? By which principle? Does it follow then that any common divisor of 215 and 258 is contained in 731? Why?

Would it be allowable, then, to drop 731 entirely, and proceed with 215 and 258? Also, in turn, to drop the second dividend 258 for the same reason?

(3) Was it shown at first that the h. c. f. of 258 and 731 could not be greater than 215? Then might it be shown in the same way that it could not be greater than 43?

---

*This topic can be omitted if desired, as few problems depend on it, and that they depend on it will be indicated in the text.

(4) Is 43 contained in 215? In 258, the sum of 215 and 43? By what principle? Is 43 contained in 516? By what principle? In 731? By what principle?

Thus, it is shown that the h. c. f. cannot be greater than 43, and that 43 is contained exactly in 258 and 731. Hence 43 is the h. c. f. sought.

Find the h. c. f. in the following by continued division:

2. 2945 and 3441. 3. 2948 and 8576. 4. 1911 and 1617.

5. 3094, 4420, 2652, 4662.

Suggestion. Find h. c. f. of first two; then find h. c. f. of result and third number; then find h. c. f. of last result and fourth number.

6. *Plan to shorten process.* In practice the process just explained can be materially shortened by removing factors from the first remainder which are evidently not common to it and either of the given numbers.

For example, in the solution to Ex. 1, the first remainder, 215, evidently contains the factor 5 while 258 does not. Hence 5 can be taken out of 215 and disregarded. Thus 43, instead of 215, becomes the divisor of 258, and is found immediately to be the h. c. f.

Again, in Ex. 2, the first remainder is 496, which contains the factor 2^4. But 2945 does not contain 2. Hence the factor 16 can be taken out of 496, leaving 31, which is then found to be the h. c. f. by dividing 2945 by it.

7. *Easy way to reduce a fraction* having large terms to its lowest terms.

Since a fraction is evidently reduced to its lowest terms by dividing numerator and denominator by the h. c. f., we have this practical rule for reducing a proper fraction to its lowest terms:

Multiply the numerator by a number which will make the product approach nearest to the denominator and find the difference between this product and the denominator. Use the remainder to find divisors of both numerator and denominator.

Reduce the following fractions to their lowest terms, or show that any fraction is already in its lowest terms:

8. $\frac{259}{481}$. Solution. $259 \times 2 - 481 = 37$. $\frac{259 \div 37}{481 \div 37} = \frac{7}{13}$. *Ans.*

9. $\frac{636}{2173}$. 10. $\frac{425}{1207}$. 11. $\frac{619}{1148}$. 12. $\frac{873}{1593}$.

13. Find the h. c. f. of $x^2 - 6x + 8$ and $4x^3 - 21x^2 + 15x + 20$.

Solution.

$$\begin{array}{rl}
4x^3 - 21x^2 + 15x + 20 & \underline{|\; x^2 - 6x + 8} \\
\underline{4x^3 - 24x^2 + 32x\phantom{{}+20}} & \underline{|\; 4x + 3} \\
3x^2 - 17x + 20 & \\
\underline{3x^2 - 18x + 24} & \\
\textit{Ans.}\ \text{h.c.f.} = \underline{x - 4}\;|\; x^2 - 6x + 8 & \underline{|\; x - 2} \\
\underline{x^2 - 4x\phantom{{}+8}} & \\
-2x + 8 & \\
\underline{-2x + 8} &
\end{array}$$

14. Find h. c. f. of $2x^2 + 11x + 5$ and $2x^2 + 9x + 4$.

15. Find h. c. f. of $6x^3 - 7ax^2 - 20a^2x$ and $3x^2 + 7ax + 4a^2$.

Solution.

$$\begin{array}{rl}
6x^3 - 7ax^2 - 20a^2x & \underline{|\; 3x^2 + 7ax + 4a^2} \\
\underline{6x^3 + 14ax^2 + 8a^2x} & \underline{|\; 2x - 7a} \\
-21ax^2 - 28a^2x & \\
\underline{-21ax^2 - 49a^2x - 28a^3} & \\
7a^2\;\underline{|\; 21a^2x + 28a^3} & \\
3x^2 + 7ax + 4a^2 & \underline{|\; 3x + 4a}\ \text{h. c. f.}\quad \textit{Ans.} \\
\underline{3x^2 + 4ax\phantom{{}+4a^2}} & \underline{|\; x + a} \\
3ax + 4a^2 & \\
\underline{3ax + 4a^2} &
\end{array}$$

Explanation. Here we take out the factor $7a^2$ from the remainder and discard it because $7a^2$ is not a factor of one of the given quantities, just as 5 could be discarded in Ex. 1. The old divisor is then divided by the other factor $3x + 4a$, which is found to be the h. c. f.

16. $6x^2 - 25x + 14$ and $2x^2 - 15x + 28$.

17. $x^3 + 7x^2 - x - 7$ and $x^3 + 5x^2 - x - 5$.

18. $7x^2 + 25x - 12$ and $17x^2 + 67x - 4$.

19. $2x^4 - 7x^3 + 5x^2$ and $x^3 + 3x^2 - 4x$.

Suggestion. First remove monomials from each quantity, reserving the common factor x as a factor of the h. c. f., the other factor being found by continued division.

20. $6x^4 - x^3 - 3x^2 - 4x - 4$ and $8x^4 - 2x^3 - 19x^2 + 3x + 10$.

II. LOWEST COMMON MULTIPLE

95. A common multiple of two or more integral quantities is an integral quantity that is exactly divisible by each of them. The **lowest common multiple** (l. c. m.) of two or more integral quantities is that quantity of smallest coefficient and lowest degree which is exactly divisible by each of them.

For example, a common multiple of $2m$, $3mn$, $6mn^2$ and $9n^3$ is $36m^2n^4$, while their l. c. m. is $18mn^3$.

Find the l. c. m. in the following and test each result to see if it contains each of the quantities and is of the lowest degree possible:

1. $4a^2,\ 6a^3b,\ 8ab^2$. *Ans.* $24a^3b^2$.
2. $3m,\ 2n,\ 5m^3n,\ 4n^2$.
3. $3a^2bc,\ 27a^3b^2c^2,\ 6ab^2c$.
4. $56a^2b^3,\ 84bx^5,\ 48x^4y^6$.
5. $a^2-16,\ a^2-2a-8$.
6. $a^3+a^2b,\ a^2-ab,\ a^2-b^2$.
7. $2ab^2c^3x^4,\ 3a^4bc^2x^2,\ 11a^3bcx$.
8. 60, 120, 48, 36.

Solution. Anything that will contain 120 will contain 60.

$120 = 2^3 \times 3 \times 5$

$48 = 2^4 \times 3$

$36 = 2^2 \times 3^2$

L. C. M. $= 2^4 \times 3^2 \times 5 = 720$. *Ans.*

Notice the *highest* power only of each factor is taken for the l. c. m.

9. 54, 81, 24, 27.
10. 40, 72, 96, 128.
11. $x^2-4y^2,\ x^2+4xy+4y^2$.
12. $x^2+2x,\ x^2+3x+2$.
13. Make a rule for finding lowest common multiple.
14. $x^2+5x+6,\ x^2+6x+8$.
15. $x^3(x-y)^2,\ y^3(x^2-y^2)$.
16. $3x^2-10xy+3y^2,\ 3x^2-4xy+y^2$.
17. $x^2-1,\ x^3+x^2+x+1,\ x^3-x^2+x-1$.
18. $6x^2-13x+6,\ 6x^2+5x-6,\ 9x^2-4$.
19. $3x^4+26x^3+35x^2,\ 6x^2+38x-28,\ 27x^3+27x^2-30x$.
20. $a^2x^2+a^2xy-abx^2-abxy,\ x^3y+2x^2y^2+xy^3,\ 3a^3b-6a^2b^2+3ab^3$.

a. In some problems it is necessary to find the highest common factor by continued division. It is then easy to write down the lowest common multiple.

III. FRACTIONS

96. A **fraction** in algebra is an expressed division. The numerator is the dividend and the denominator is the divisor.

97. Reduction of Fractions. By **reduction** in arithmetic and algebra is meant changing the form of a quantity without altering its value.

98. Fundamental Principle in Fractions. *Both terms of a fraction may be multiplied or divided by the same quantity without altering its value.*

PROOF. Let $\frac{a}{b}$ = any fraction, r = its value, and m = any number.

Then, $a = br$. (Since the dividend equals the quotient × divisor, § 46.)

also, $ma = mbr$. (By the multiplication Axiom, § 57.)

or, $ma = r \cdot mb$. (By the commutative and associative laws of multiplication, § 39.)

Therefore, $\frac{ma}{mb} = r$. (By the Division Axiom, dividing through by mb.)

Or, $\frac{ma}{mb} = \frac{a}{b}$. (Numbers equal to same number are equal to each other, Axiom 8, § 57.)

It should be observed that since a and b may represent any numbers whatever, whether integral, fractional, ordinary irrational, as $\sqrt{3}$, or other, we *assume* they have a definite quotient, and assign r to take the place of this quotient. When we multiply through by m, we *assume* that the multiplication axiom holds true for all values of m and the quantities into which it is multiplied.

99. Exercise in reducing Fractions to their Lowest Terms. Factor both numerator and denominator, and cancel common factors by the principle just given.

Reduce the following fractions to their lowest terms:

1. $\frac{x^4y^6z^2}{x^6y^2z^2}$. 2. $\frac{12a^7b^2}{15a^2b^3}$. 3. $\frac{4x-16}{x^2-16}$. 4. $\frac{x^5-a^2x^3}{x^4-a^4}$.

5. $\dfrac{x^3-27}{x^2-2x-3}$. 6. $\dfrac{x^2-1}{2xy+2y}$. 7. $\dfrac{a^n}{a^{n+1}}$. 8. $\dfrac{a^2-a-20}{a^2+a-12}$.

In some problems it is an advantage to use the continued division method to find the factor by which to divide both terms.

9. $\dfrac{x^3-9x^2+26x-24}{x^3-10x^2+31x-30}$. 10. $\dfrac{6x^3+13x^2+15x-25}{2x^3+4x^2+4x-10}$.

100. Allowable Changes in the Signs of a Fraction. There are *three signs* connected with every fraction; that of the numerator, that of the denominator, and that of the quotient, which stands before the fraction and is called the sign of the fraction.

Using a and b as heretofore to denote any quantities, whether monomials or polynomials, we have

$$+\left(\frac{+a}{+b}\right)=+\left(\frac{-a}{-b}\right)=-\left(\frac{-a}{+b}\right)=-\left(\frac{+a}{-b}\right),$$

since each of these expressions has the value $\frac{a}{b}$. To understand clearly why this is so one must remember that like signs in dividend and divisor give a positive quotient and unlike signs give a negative quotient, and that a $-$ sign before a parenthesis changes the sign of the quantity within when the parenthesis is removed, and that a $+$ sign does not.

Similarly, $$+\left(\frac{-a}{+b}\right)=-\left(\frac{+a}{+b}\right)=+\left(\frac{+a}{-b}\right)=-\left(\frac{-a}{-b}\right),$$

since each of these expressions is equal to $-\frac{a}{b}$.

By examining the preceding results we learn that:

Changing any two of the signs of a fraction does not alter its value, while changing one or all three changes the value of the fraction.

The student will do well, in changing the sign of the numerator or denominator when they are polynomials, to multiply by -1.

The pupil may now explain the following changes:

1. $-\dfrac{-a+b+c}{d}=\dfrac{a-b-c}{d}$; $-\dfrac{3y-x}{m+n}=\dfrac{x-3y}{m+n}$.

a. Notice that $a-b-c=-1\times(b+c-a)$. If two polynomials have the same terms but each term with the opposite sign, one polynomial is the negative of the other. Thus $-(3y-x)=x-3y$.

2. $-\dfrac{a+b-c}{c-b}=\dfrac{a+b-c}{b-c}$; $-\dfrac{c}{d-e+f}=\dfrac{c}{e-d-f}$.

3. $\dfrac{-m+n}{-p-q}=\dfrac{m-n}{p+q}$; $\dfrac{-3x-2y+z}{y+z-x}=\dfrac{3x+2y-z}{x-y-z}$.

101. Signs of Factors of Numerator or Denominator Changed. The signs of an even number of the *factors* of the numerator or denominator can be changed without altering the value of the fraction, while changing the signs of an odd number of such factors changes the sign of the fraction. This is because an even number of -1 factors multiplied or divided gives $+1$, while an odd number gives -1.

The student may explain the following:

1. $\dfrac{abc}{(a-b)(c-b)}=-\dfrac{abc}{(a-b)(b-c)}$; $\dfrac{m}{n-m}=-\dfrac{m}{m-n}$.

2. $\dfrac{(y-x)(n-m)}{(b-a)(d-c)}=\dfrac{(x-y)(m-n)}{(a-b)(c-d)}$;

$\dfrac{(n-m)(q-p)}{d-c}=-\dfrac{(m-n)(p-q)}{(c-d)}$.

The student should notice carefully the difference between a **term** of the numerator or denominator and a **factor** of either. Let him define each word. A term can be removed (destroyed) only by addition or subtraction; a factor can be removed (canceled) only by division of both numerator and denominator by the same quantity.

102. A **mixed quantity** is composed of an integral quantity and a fraction. In algebra the sign + or − always appears between the integral quantity and the fraction, since otherwise the operation of multiplication would be indicated (§ 4). Hence, in algebra, reduction of a mixed quantity to an improper fraction is merely a case in addition of fractions.

103. Addition and Subtraction of Fractions.

Type form: $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}.$

1. $\frac{2}{3} + \frac{5}{6} + \frac{7}{8} + \frac{15}{16}.$ 2. $5\frac{2}{3} + 7\frac{3}{10} - 2\frac{7}{15} + 4\frac{7}{20}.$

3. $\frac{x}{x+1} - \frac{x^2 - 3x + 2}{x^2 - 1} + 3.$ Check with $x = 2.$

Solution. L.c.d. (lowest common denominator) is $x^2 - 1$. The denominator of the integral quantity, 3, is 1 understood.

$$\frac{x(x-1)}{x^2-1} - \frac{x^2-3x+2}{x^2-1} + \frac{3(x^2-1)}{x^2-1}.$$

(By dividing l.c.d. by the denominator of each fraction, and then multiplying both terms by the quotient, § 98.)

$$\frac{x^2 - x - (x^2 - 3x + 2) + 3x^2 - 3}{x^2 - 1}.$$

(By indicating the addition of the numerators and placing the sum over the common denominator, § 35.)

$$\frac{3x^2 + 2x - 5}{x^2 - 1} = \frac{(3x+5)(x-1)}{x^2-1} = \frac{3x+5}{x+1}. \quad \textit{Ans.}$$

(Simplifying and reducing to lowest terms.)

Check. Let $x = 2$.

Given quantity $= \frac{x}{x+1} - \frac{x^2 - 3x + 2}{x^2 - 1} + 3 = \frac{2}{3} - \frac{4 - 6 + 2}{4 - 1} + 3 = 3\frac{2}{3}.$

Answer $= \frac{3x+5}{x+1} = \frac{3 \times 2 + 5}{2 + 1} = 3\frac{2}{3}.$

a. In checking, values which make any denominator 0 must be avoided. Thus, $x = 1$ could not have been used in preceding problem.

4. $\frac{a+x}{a-x} + \frac{a-x}{a+x} + \frac{4ax}{a^2 - x^2}.$ 5. $m - \frac{m^2 + n^2}{m - n} + n.$

6. $\frac{a}{a-2} - \frac{a-2}{a+2} + \frac{3}{4 - a^2}.$

Suggestion. Change last fraction to $-\frac{3}{a^2 - 4}.$ (§ 100.)

7. $a + b - \frac{a^2 - 2ab + b^2}{a + b}.$ 8. $\frac{x^2 + ax + a^2}{x^3 - a^3} - \frac{x^2 - ax + a^2}{x^3 + a^3}.$

9. $\frac{2}{x^2 - 3x + 2} + \frac{2}{x^2 - x - 2} - \frac{1}{x^2 - 1}.$

10. $\frac{a}{a-b} + \frac{b}{b-c} + \frac{c}{c-d}.$

104. Reduction of Improper Fractions to Mixed or Integral Quantities. An **improper fraction** in algebra is one that can be reduced to a mixed or integral quantity. A **proper fraction** is one that cannot be so reduced. To reduce an improper fraction to a whole or mixed quantity, divide the numerator by the denominator, either by short or long division, and add the remainder, if any, over the divisor to the integral quantity.

1. $\frac{ax+2x^2}{a+x}$. *Ans.* $x+\frac{x^2}{a+x}$. 2. $\frac{x^2+2x-15}{x-3}$.

3. $\frac{4x^2-2x}{2x^2-x+1}$. 4. $\frac{a^5x-3ax^5}{a^3-ax^2}$.

105. Multiplication and Division of Fractions.

Type forms: $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$; $\frac{m}{n}\div\frac{p}{q}=\frac{m}{n}\times\frac{q}{p}=\frac{mq}{np}$.

1. $\frac{a^2x-x^3}{a^2-121}\times\frac{a+11}{2ax-2x^2}$.

SOLUTION. $\frac{\cancel{x}(a+x)\cancel{(a-x)}}{\cancel{(a+11)}(a-11)}\times\frac{\cancel{a+11}}{2\cancel{x}\cancel{(a-x)}}=\frac{a+x}{2(a-11)}$. *Ans.*

CHECK. Let $a=2$, $x=1$.

Then, given quantity $=\frac{3}{-117}\times\frac{13}{2}=-\frac{1}{6}$;

also, answer $=\frac{3}{-18}=-\frac{1}{6}$.

2. $\frac{a^2-121}{a^2-4}\times\frac{a+2}{a+11}$. 3. $\frac{a+x}{a}\times\frac{a-x}{x}\times\frac{a^2+x^2}{a^2-x^2}$.

4. $\frac{a^2x-x^3}{a}\times\frac{3a}{2ax-2x^2}$. 5. $\frac{x-3}{x+1}\times\frac{x^2+2x+1}{x^3-27}$.

6. $\frac{2a^3-2ab^2}{a+2b}\div\frac{a^2-b^2}{2a+4b}$. 7. $\frac{8a^5b^6x^7}{15c^6y^5}\times\frac{5a^4b^3c^2}{6x^8y^3}\times\frac{3c^5y^4}{4a^{15}b^4}$.

8. $\frac{45dx-9dy}{20abx-10b^2x}\div\frac{30x-6y}{20a^2x-5b^2x}$.

9. $-\frac{x^2+5x+6}{x^2-1}\div-\frac{x^2-9}{x^2-2x-3}$.

10. $\left(\frac{x^2}{y^3}-\frac{1}{x}\right)\div\left(\frac{x}{y^2}+\frac{1}{y}+\frac{1}{x}\right).$

SUGGESTION. In all such problems reduce both dividend and divisor to simple fractions by § 103.

11. $\left(x^3-\frac{1}{x^3}\right)\div\left(x-\frac{1}{x}\right).$ 12. $\left(x-\frac{3x}{1+x}\right)\div\left(\frac{x^2-2x}{1-x}\right).$

13. $\left[\frac{x^2-64}{x^2+24x+128}\times\frac{x^2+12x-64}{x^3-64}\right]\div\frac{x^2-16x+64}{x^2+4x+16}.$

14. $\left\{\frac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2}\div\frac{3ab-3b^2}{4}\right\}\div\frac{a^2+ab}{2a-2b}.$

106. Complex Fractions. A complex fraction is a fraction that has a fraction in one or both of its terms. It is essentially a problem in division of fractions, and is simplified by dividing the numerator by the denominator.

1. $\dfrac{\dfrac{3cy}{5a^2x}}{\dfrac{5b}{7axy}}.$ *Ans.* $\dfrac{21cy^2}{25ab}.$ 2. $\dfrac{\dfrac{3a^2}{4b}}{\dfrac{3a}{7b^2}}.$ 3. $\dfrac{\dfrac{a+x}{2x}}{\dfrac{y-a}{x}}.$

4. $\dfrac{x-1+\dfrac{6}{x-6}}{x-2+\dfrac{3}{x-6}}.$ SOL. $\dfrac{\dfrac{x^2-7x+12}{x-6}}{\dfrac{x^2-8x+15}{x-6}}=\dfrac{(x-3)(x-4)}{x-6}\times\dfrac{x-6}{(x-3)(x-5)}$

$=\dfrac{x-4}{x-5}.$ *Ans.*

CHECK. Let $x=7$. Then given fraction $=\frac{12}{8}$; also, answer $=\frac{3}{2}$.

a. A good plan to simplify many complex fractions is to multiply both terms by the l. c. d. of the denominators.

Thus, both terms of Ex. 4 could have been multiplied by $x-6$, which would have given $\frac{x^2-7x+12}{x^2-8x+15}$ directly.

5. $\dfrac{x+\dfrac{2x}{x-3}}{x-\dfrac{2x}{x-3}}.$ 6. $\dfrac{\dfrac{1}{x}+\dfrac{4}{x^2}+\dfrac{4}{x^3}}{1+\dfrac{5}{x}+\dfrac{6}{x^2}}.$

107. Powers of Fractions. To raise a fraction to a power, both of its terms, of course, are raised to this power.

1. $\left(\dfrac{x^2-1}{x^2-x}\right)^3.$

SOLUTION. $\left(\dfrac{(x+1)(x-1)}{x(x-1)}\right)^3=\dfrac{(x+1)^3(x-1)^3}{x^3(x-1)^3}=\dfrac{(x+1)^3}{x^3}.$ *Ans.*

2. $\dfrac{(x^2-5x+6)^3}{(x-2)^2}.$ 3. $\dfrac{(x^4-xy^3)^{15}}{(x^3-y^3)^{12}}.$ 4. $\dfrac{(x^3-1)^3}{2(x^2+x+1)^2}.$

108. Some Common Errors in Work with Fractions. Certain errors are made so often and by so many students that it is desirable to describe them and caution against them. Checking serves to find such errors.

1. Never cancel single *terms* in the numerator and denominator of a fraction when either is a polynomial. Only *factors* of the whole numerator or whole denominator can be canceled by the fundamental principle (§ 98). Hence, to simplify a simple fraction, cancel factors in both numerator and denominator.

2. In subtraction of fractions, and later in equations, when a fraction is preceded by $-$, students very often *forget* to change the signs of all the terms of the numerator in simplifying. Because of the danger of forgetting, the use of parentheses is advised for all who are not expert in performing this operation. Thus,

$$\frac{a}{b}-\frac{c-d}{b}=\frac{a-(c-d)}{b}=\frac{a-c+d}{b}.$$

3. To change the signs of both numerator and denominator, the signs of all the terms of each must be changed.

Thus, $\dfrac{2m-n}{3p-2q}=\dfrac{n-2m}{2q-3p}.$ This amounts to multiplying both terms by -1, by § 98. See § 100.

4. To multiply a fraction by an integral quantity, multiply the numerator by the integral quantity and place result over the denominator. Do *not* multiply both terms by the integral quantity,

since by the fundamental principle this will leave the multiplicand unaltered. There is much less liability of mistakes if 1 is written for the denominator of integral quantities in all four of the fundamental operations in fractions.

5. In division of fractions both dividend and divisor must be either simple fractions or integral quantities. If they are not such, they must be reduced by performing the operations indicated. (See § 105, 10.)

6. In addition of fractions, after the several fractions are reduced to a common denominator and the numerators are added, the sum *must* be placed over the common denominator. Students are particularly likely to forget this when coming back to fractions after studying equations. In the solution of equations, denominators are dropped, the multiplication axiom having been used to make them disappear.

109. Type Forms in Fractions.

1. $\frac{a}{b}=\frac{ma}{mb}$. 2. $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$. 3. $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$.

4. $a+\frac{b}{c}=\frac{ac+b}{c}$. 5. $a-\frac{b}{c}=\frac{ac-b}{c}$. 6. $\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd}$.

7. $\frac{a}{b}\times c=\frac{ac}{b}$. 8. $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$. 9. $\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$.

10. $\dfrac{\frac{a}{b}}{c}=\frac{a}{b}\div\frac{c}{1}=\frac{a}{bc}$. 11. $\dfrac{a}{\frac{b}{c}}=\frac{a}{1}\div\frac{b}{c}=\frac{ac}{b}$.

12. $\dfrac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$.

110. Miscellaneous Exercise in Fractions. See § 108 for points to be kept in mind.

Simplify the following:

1. $\dfrac{mx-nx}{(a+b)(m-n)}$. 2. $a+x-\dfrac{4ax-5x^2}{a-x}$.

3. $\dfrac{x^2-5x+6}{x^2-7x+12}.$

4. $\dfrac{1}{D-2}+\dfrac{2}{(D-2)^2}.$

5. $\dfrac{a^2b^2+3ab}{4a^2-1}\div\dfrac{ab+3}{2a+1}.$

6. $\dfrac{3-x}{1-3x}-\dfrac{3+x}{1+3x}-\dfrac{1-16x}{9x^2-1}.$

7. $\left(1-\dfrac{a^4}{b^4}\right)\div\dfrac{a^2+b^2}{ab}.$

8. $\dfrac{2}{x}-\dfrac{3}{2x-1}-\dfrac{2x-3}{4x^2-1}.$

9. $(1-G^2)\div\dfrac{(1-G)^2}{G}.$

10. $\dfrac{1}{R-1}-\dfrac{1}{R+2}-\dfrac{3}{(R+2)^2}.$

11. $\dfrac{ax-x^2}{(a+x)^2}\div\dfrac{x^2}{a^2-x^2}.$

12. $\left(x^2+\dfrac{1}{x^2}+2\right)\div\left(x+\dfrac{1}{x}\right).$

13. $\dfrac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}.$

14. $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}.$

15. $\dfrac{x^2+xy}{x^2+y^2}\times\dfrac{x^3-y^3}{xy(x+y)}.$

16. $\left(\dfrac{x^2}{a^2}-\dfrac{x}{a}+1\right)\left(\dfrac{x^2}{a^2}+\dfrac{x}{a}+1\right).$

17. $\left(x^4-\dfrac{1}{x^4}\right)\div\left(x+\dfrac{1}{x}\right).$

18. $\dfrac{3}{x-3}-\dfrac{2}{x-4}-\dfrac{x-6}{(x-2)(x-5)}.$

19. $\dfrac{R^2-r^2}{t^2}\times\dfrac{t^2-t}{R^2-2Rr+r^2}.$

20. $\dfrac{a^{x-2}b^{x+1}c^x}{x^{a-b}y^cz^{a-1}}\times\dfrac{x^{a+b}y^bz^3}{a^2b^{x+1}c^3}.$

21. $\dfrac{1}{1+\dfrac{q+1}{n-q}}.$

22. $\dfrac{2a}{a^4+a^2+1}-\dfrac{1}{a^2-a+1}+\dfrac{1}{a^2+a+1}.$

23. $\left(4x^2+14x+\dfrac{98x-27}{2x-7}\right)\left(\dfrac{1}{6}-\dfrac{3x+29}{12x^2+18x+27}\right).$

24. $\dfrac{4a(a^2-x^2)}{3b(c^2-x^2)}\div\left[\dfrac{a^2-ax}{bc+bx}\times\dfrac{(a+x)^2}{(c-x)^2}\right].$

25. $\dfrac{2a}{(x-2a)^2}-\dfrac{x-a}{x^2-5ax+6a^2}+\dfrac{2}{x-3a}.$

26. $\dfrac{1}{b}\left(\dfrac{1}{a-b}-\dfrac{1}{a+2b}\right)-\dfrac{2}{a^2+ab-2b^2}.$

27. $\left[\dfrac{x^4-a^4}{x^2-2ax+a^2}\div\dfrac{a^2+ax}{x-a}\right]\div\dfrac{x^5-a^2x^3}{x^3+a^3}.$

CHAPTER IV

GRAPHS—RATIO AND PROPORTION—FUNCTIONALITY

I. GRAPHS

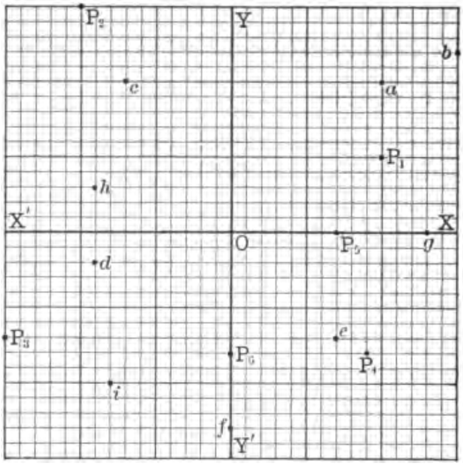
111. Graph Paper. Unit of Measure. Graph paper is paper very accurately ruled into little squares or parallelograms. Most graph paper has the centimeter (0.4 in. nearly) as the unit of measure, and divides each square centimeter into twenty-five little squares. Since 10 mm. make 1 cm., one side of the little square is thus 2 mm. in length.

112. Location of Places by reference to Standard Lines. In graphs the position of a point is located very much as street corners or section corners usually are, as so many blocks or sections north or south, and so many blocks or sections east or west of two streets or roads passing through a central square of a town.

113. Axes. Two centimeter lines on the graph paper are made heavier than the others and designated as the *axes*; the horizontal one as the X axis, and the vertical one as the Y axis. The point where these axes cross is called the *origin.* Other points are located by saying they are so many units (centimeters or millimeters) to the right or left of the Y axis, and so many units above or below the X axis. On the graph diagram distances measured to the right from the Y axis are positive, and those to the left are negative; those measured upward from the X axis are positive, and those downward are negative.

114. Location of Points. The distance to the right or left from the axis of Y to a point is called its **abscissa**, or x distance; the distance from the X axis to the point is called its **ordinate**, or y distance. The abscissa and ordinate of a point taken together are called the **coördinates** of the point.

1. In the figure, the abscissa of P_1 is $+2$ and the ordinate $+1$; the coördinates of P_2 are $x = -2$, $y = 3$; the coördinates of P_3 are $x = -3$, $y = -1.4$; of P_4 are $x = 1.8, y = -1.6$; of P_5 are $x = 1.4$, $y = 0$; of P_6 are $x = 0$, $y = -1.6$. For the rules for signs used see the preceding article.



2. Write the coördinates of the points located by the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, on the diagram.

3. On a piece of squared paper draw two heavy lines over two centimeter lines for axes as in the diagram just given. Now locate the following points each by a dot on the diagram, writing *a* beside the dot of the first point located, *b* beside the second, and so on.

a. $(x = 3, y = 1)$.
b. $(x = +3, y = +3)$.
c. $(x = -2, y = 1)$.
d. $(x = -1, y = 2)$.
e. $(x = 2.4, y = -1.6)$.
f. $(x = -1.2, y = -1.8)$.
g. $(x = -.6, y = 2.2)$.
h. $(x = 0, y = -2.4)$.
i. $(x = -5, y = 0)$.
j. $(x = -4.1, y = -1.6)$.

4. Make another diagram with axes and locate the following points, understanding that the first number inside the parenthesis gives the value of x or the abscissa, and the second number the value of y or the ordinate.

a. $(2, 2)$.
b. $(2, 6)$.
c. $(-3, 4.6)$.
d. $(-2, -4.8)$.
e. $(-3, 7.4)$.
f. $(-3, -8.1)$.
g. $(-4.9, 0)$.
h. $(7.3, -2.6)$.
i. $(0, 0)$.
j. $(0, -6.5)$.
k. $(12, -14)$.
l. $(5.5, 0)$.

115. Graphs. If a series of points is "plotted" (that is, located and marked on a diagram), representing values of a quantity y that changes as another quantity x changes, and these points are joined by a running line, this line is called the **graph** of the law or data that determined the points.

In § 60 a number of graphs were constructed from statistical data. Such data can come from a great variety of sources. Thus, writers on history and economics use graphs to show to the eye quickly the changes in population, expenditures, and production. Scientists use them to show the laws of nature, engineers to show the working of machinery, and business houses to show the changes in prices, cost of production, sales, etc. In short, graphs have a wide range of uses, and the student should learn to construct and read them readily. In science and engineering graphs are often constructed mechanically, as by the thermograph, barograph, anemograph, etc.

1. **Statistical Graph.** The table gives the number of survivors at different ages out of 100,000 particular persons alive at age ten:

Age	Survivors	Age	Survivors	Age	Survivors
10	100,000	40	78,106	70	38,569
15	96,285	45	74,173	75	26,237
20	92,637	50	69,804	80	14,474
25	89,032	55	64,563	85	5,485
30	85,441	60	57,917	90	847
35	81,822	65	49,341	95	3

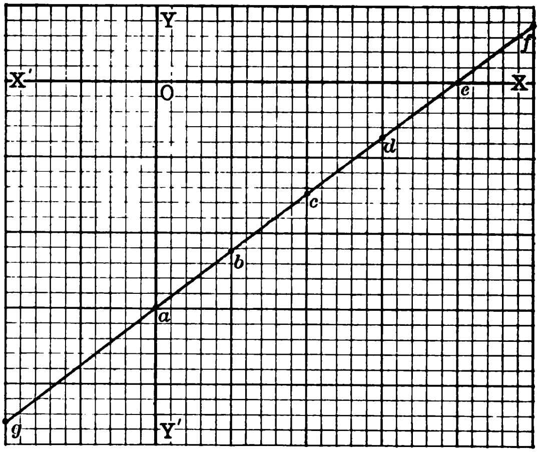
Let 1 cm. on Y axis represent 10,000 persons, and 1 cm. on X axis ten years. Then coördinates of first point are $x = 1$, $y = 10$.

2. **Graph of an Equation.** In the equation, $y = 3x + 5$, we see that y is a function of x, that is, depends on x for its value. As x changes in value, y also changes correspondingly.

We see that for every value we assign to x there will be a corresponding value for y. Such sets of corresponding values of x and y can be taken for the coördinates of points. If these points are connected with a running line, we have a graph. Evidently any equation, as $y = x^2 + 4x - 6$, will have a graph to correspond to it.

116. Graphs of Simple Equations or Equations of the First Degree (§ 90).

1. Construct the graph of the equation $3x - 4y = 12$.



TABLE

$3x - 4y = 12$

(x, y)	Pt.
$(0, -3)$	a
$(1, -2.25)$	b
$(2, -1.5)$	c
$(3, -.75)$	d
$(4, 0)$	e
$(5, .75)$	f
$(-2, -4.5)$	g

EXPLANATION of how the values in the table are found. If $x = 0$ in the given equation, $3x - 4y = 12$, the equation reduces to $-4y = 12$, whence $y = -3$, as given in the table opposite a. If $x = 1$, $3x - 4y = 12$ becomes $3 - 4y = 12$, whence, transposing 3, $-4y = 9$, whence, $y = -2.25$, which is given opposite b; and in the same manner the other values of y are found. We simply *take* any convenient values for x and find the corresponding values of y, and write the pairs of corresponding values in parentheses for the coördinates of points.

Having found the coördinates of a series of points, a, b, c, d, e, f, g, we now locate these points on a diagram. The point a, whose coördinates are $(0, -3)$, is located first by a dot with the letter a beside it; then the point b is located in the same way; and so on. Last of all a line is drawn through these points.

The student may now construct in the same way the graphs of the following equations. He can let $x = 0, 1, 2, 3, 4, 5, -2$ as in the preceding example.

2. $3x - 2y = 5$. 3. $3x + 5y = 15$. 4. $2x - 7y = -12$.

5. *The graph of a simple equation, or an equation of the first degree, will be found to be always a straight line,* as in Ex. 1–4.

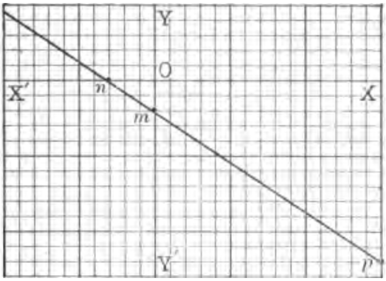
This being the case, since *two* points determine a straight line, it will be necessary to locate only two points to locate the graph on the diagram. Now the easiest pair of coördinates to obtain are those found by first putting $x = 0$, and getting the corresponding value of y, and then putting $y = 0$, and getting the corresponding value of x.

To check the correctness of the location of the graph, it is desirable to locate a third point near the margin of the sheet. Now, whenever the two points first located fall close together, a slight inaccuracy in plotting either of them will throw the extremities of the graph very considerably out of the correct position. In such instances the graph should be constructed by joining the marginal point to the more remote of the first two. The other point is then used only as a check. Calculate decimals usually to tenths only.

6. Construct the graph of $10x + 15y = -6$.

$10x + 15y = -6$

(x, y)	PT.
$(0, -.4)$	m
$(-.6, 0)$	n
$(3, -2.4)$	p



Here, because m and n are close together, we join n and p to get the graph, and check with m.

Construct the graphs of the following equations, following the model just given:

7. $5x + 3y = -10.$

8. $-2x + 5y = 12.$

9. $6x - 7y = 1.$

10. $25x + 7y = 29.$

11. $6x + 5y = 16.$

12. $35x + 17y = -86.$

13. $x = 5.$

SUGGESTION. y can have *any* value. Locate two points. The graph is a line parallel to the Y axis, 5 units to right of YY'.

14. $y = -3$.   15. $\frac{x}{2} - \frac{y}{3} = 4.5$.   16. $\frac{x+3y}{2} = \frac{y}{5} - 1$.

17. $2x = 5y$.

SUGGESTION. Putting $x = 0$ or $y = 0$ gives the origin. To find a second point, another value must be assigned to x or y.

18. $3x - 11y = 0$.   19. $x = y$.   20. $x = -y$.

21. $x + 2y = 75$.

SUGGESTION. In such a problem the centimeter as unit runs the points located off the diagram. Instead of the centimeter use the millimeter or 2-millimeter length as unit.

22. $3x - y = 120$.   23. $\frac{x}{13} - \frac{y}{7} = 6x - 10y - 8$.

II. RATIO AND PROPORTION

117. The **ratio** of two quantities is the *quotient* obtained by dividing the first by the second. A ratio is thus a fraction. It is usually denoted by a colon, but preferably by a fraction. Thus, $2:3$ or $\frac{2}{3}$ is the ratio of 2 to 3, the sign : having the same meaning as $\div$.

1. What is the ratio of 9 to 12? Of a to b? Of $a + b$ to $c + d$? Of $a^2 - b^2$ to $(a + b)^2$? Of $\frac{7}{12}$ to $\frac{5}{7}$? Of a to $b + c$?

2. The first number in a ratio is called its **antecedent**, and the second its **consequent**; both are called the **terms of the ratio**. A ratio is always an abstract number, whatever concrete numbers may be in its terms, since it is always the quotient of one number divided by another of the same kind.

118. A **proportion** is an equation (§ 54) whose members are ratios.

For example, $\frac{6}{9} = \frac{10}{15}$; $6:9 = 10:15$; $\frac{a}{b} = \frac{c}{d}$ are proportions.

By definition four terms are in proportion when the first term divided by the second equals the third term divided by the fourth.

119. Fundamental Theorem in Proportion. *In a proportion, the product of the first and last terms, called the "extremes," equals the product of the two middle terms, or "means."*

PROOF. Let $\frac{a}{b}=\frac{c}{d}$ be any proportion.

Then, $\frac{a}{b}\times bd=\frac{c}{d}\times bd$, (By the multiplication axiom, § 57.)

or $ad=bc$. (By rule for multiplication of fractions. Notice a and d are the extremes, and b and c, the means.)

The importance of this theorem consists in the fact that it changes a proportion into an *integral equation*.

120. Exercise in Proportion. Using the definition, § 118, to solve Ex. 1, and the theorem of § 119 to solve Exs. 2, 3, find the unknown in the following proportions:

1. $\frac{8}{16}=\frac{36}{x}$; $\frac{x}{96}=\frac{7}{42}$; $100:x=50:75$; $45:60=x:24$.

2. $8:9=10:x$; $\frac{2}{3}:\frac{1}{2}=x:7$; x da. : 7 da. $=21:16$.

3. Solve $\frac{a}{b}=\frac{c}{d}$, or $a:b=c:d$, for a, b, c, and d in turn.

4. The answers to Ex. 3 can be written $a=b\times\frac{c}{d}$; $b=a\times\frac{d}{c}$; $c=d\times\frac{a}{b}$; $d=c\times\frac{b}{a}$. Now, $\frac{a}{b}=\frac{c}{d}$ is a formula for every simple proportion. We see, then, that the unknown is always equal to the term corresponding to it multiplied by the proper ratio of the other two terms. Hence we have the following:

5. **Rule** for solving any arithmetical problem in proportion whose terms are concrete numbers. *Write first the number corresponding to the answer with its name; multiply this by the improper fraction ratio of the remaining two terms if the answer is to be greater than the first factor, and by the proper fraction ratio, if it is to be less.*

Always reduce the ratio factor to its lowest terms before writing it down, and cancel to the best advantage before multiplying.

This is the business man's rule for solving any analysis or proportion problem, and should be mastered by the student. It is the easiest and quickest way available.

Solve the following problems by the rule just given and check the result by an analysis solution.

6. If 45 yd. of cloth cost $72, what will be the cost of 65 yd.?

SOLUTION. $72 × $\frac{13}{9}$ = $104. *Ans.*

EXPLANATION. $72 corresponds to the answer. Answer is to be more than $72, so ratio is improper fraction, viz. $\frac{65}{45}$.

CHECK. If 45 yd. cost $72, 1 yd. costs $1.60; then 65 yd. cost 65 × $1.60, or $104.

7. If a building known to be 75 ft. high casts a shadow 125 ft. long, how high is a tree that casts a shadow 105 ft. long?

8. If an express train travels 375 mi. in 10 hr., what time would it require to go 525 mi.?

9. 26 men build a wall in 21 da.; how many men could have built it in 14 days?

10. A water tower 100 ft. high and 22 ft. in diameter is to be represented in a drawing as 8 in. high. How many inches shall there be in the diameter of the drawing?

11. The height of the mercury in a barometer is proportional to the pressure. If the pressure is 14.7 lb. to the square inch when the mercury is 30 in. high, what is the pressure when the mercury is only 28 in. high?

12. If the wages of 48 men for 11 days of 9 hr. each is $1276, what would be the wages of 27 men for 13 days of 8 hr. each?

SOLUTION. $1276 × $\frac{9}{16}$ × $\frac{13}{11}$ × $\frac{8}{9}$ = $754. *Ans.*

EXPLANATION. Each ratio is found independently of the others. Thus, if a certain sum is earned in 11 da., in 13 da. there will be earned $\frac{13}{11}$ times as much.

CHECK. A "cause and effect" solution is much preferable here to an analysis one as a check. By this method the two causes, or the products of the numbers of men, days, and hours, and the two effects, dollars, form in order the four terms of a simple proportion, x taking the place of the unknown. Evidently a "man-hour," or what a man can do in an hour, is the unit used in the causes.

13. If 6000 ft. of copper wire $\frac{1}{16}$ in. in diameter costs $10 when copper is 15¢ a pound, what will be the cost of 3850 ft. of $\frac{3}{32}$ in. wire when copper is 14¢ a pound?

SUGGESTION. The cost will be in proportion to the *areas* of the cross sections, or to the squares of the diameters $\frac{1}{16}$ and $\frac{3}{32}$.

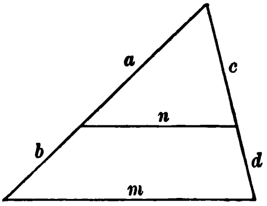
14. Three men go into partnership, the first furnishing \$ 6000 of the capital, the second \$ 5000, and the third \$ 4000. They clear the first year \$ 2400. What is each man's share of the profits? Show how to solve this problem by the rule in 5 above.

15. A bankrupt owes three creditors \$ 240, \$ 360, and \$ 400 respectively. If \$ 325 is divided among them in proportion to their claims, what will each one get?

16. If silver solder contains 1 part brass, 1 part copper, and 19 parts silver, how much of each would there be in 20 oz. of solder?

17. A substance by analysis was found to contain 8.08 units (by weight) of sodium, 5.61 units of sulphur, and 11.31 units of oxygen. How many pounds of sulphur are in 11.4 lb. of the substance?

18. If a line n is drawn parallel to the base m of a triangle, dividing the other two sides into the four segments a, b, c, d, make all the proportions you can out of these six letters. See § 51. Find two different expressions for the value of a. Also two for m.



19. If two chords intersect in a circle, make a proportion out of the two segments of the first chord respectively, a and b, and the two segments of the second chord respectively, c and d. If $a = 4$, $b = 6$, $d = 11$, find c. See § 51.

20. If two secants a and b intersect without a circle, make a proportion out of them and their external segments which are respectively c and d. Find c when $a = 10$, $b = 12$, $d = 4$.

21. If a perpendicular p is dropped from the vertex of a right triangle on the hypotenuse h, dividing it into the segments m and n lying respectively next to the other two sides of the triangle, a and b, what proportions hold between these letters? Solve for m

and n, getting two different expressions as values for m. Find m when $a = 6.5$ and $h = 25$.

22. If t is a tangent to a circle from a point outside and s is a secant from the same point and e is its external segment, find e when $t = 6.4$ and $s = 13$.

121. Study of Proportion in which two terms are given, the third term has various values *assigned* to it, and the corresponding values of the fourth are to be found.

A typical proportion problem has *two* kinds of quantities involved, each having two values; as, if 5 lb. of nails cost 28¢, what will 7 lb. cost? We readily see that if we know that 5 lb. cost 28¢, we can *take* any other number of pounds and find the cost, or any cost and find the corresponding number of pounds. This truth will be used in the next article.

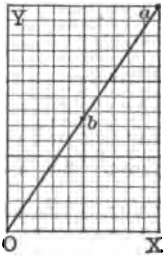
122. Proportions and their Graphs. For a graph we need two varying quantities, x and y. In a proportion there are four terms of which x and y can be two and the other two must be known numbers.

1. To find the graph of the proportion $x : y = 2 : 3$, or $y = \frac{3}{2}x$.

SOLUTION:

$$y = \tfrac{3}{2}x$$

(x, y)	PT.
(0, 0)	0
(2, 3)	a
(1, 1.5)	b

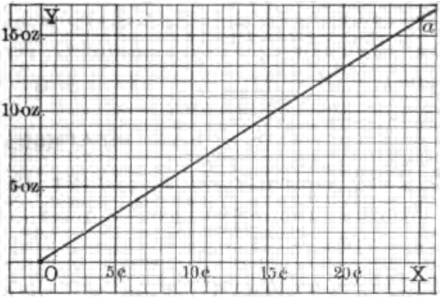


We see from this that the graph of a proportion, of which x and y are the terms of one ratio, is *a straight line through the origin* and through a point whose coördinates are the terms of the given ratio, the term corresponding to x being the *abscissa* and that corresponding to y the ordinate.

Since three terms are always given in a proportion to find the fourth, we must *assign* values to either x or y, and the graph will then give a corresponding value for the other letter. Thus, if $y = 2$ in the graph, the corresponding value of x is found by running along the graph $y = 2$ (or the line parallel to x and 2 units above it) until we come to graph Oa, and thence down vertically to OX, where we get the corresponding value of x, or 1.3^+. Or, again, if $x = .8$, y is found to be 1.2 by passing around in the opposite way.

2. Construct a cost graph when 1 lb., or 16 oz., costs 25¢.

Solution. The graph passes, of course, through the origin, and is made to go through the point (25, 16). From it can be read off the cost of any number of ounces by starting at the number of ounces above O, going thence on a horizontal line to the graph, thence downward in a vertical line to OX, where the cost can be read off from the OX scale of numbers.



Similarly the number of ounces corresponding to any given cost can be found by starting at the cost point on OX, going thence on a vertical line upward to the graph, thence on a horizontal line to OY, where the number of ounces can be read off from the OY scale.

3. Using the graph to Ex. 2, find the cost of 5, 8, 9, 17 oz., respectively, checking each time with an arithmetical solution. Find, also, the number of ounces corresponding to 12, 18, 23, 4 cents, respectively, and check with arithmetical solutions.

4. Solve graphically Ex. 6–9, § 120.

5. Make a graph to reduce from pounds to kilograms, and conversely, knowing that 1 kg. = 2.2 lb. (nearly); also one to reduce from miles to kilometers, and conversely, knowing that 1 mi. = 1.6 km. (nearly); also one to reduce from meters to yards, and conversely, knowing that 1 m. = 1.1 yd. (nearly). (These figures are approximations, correct to one decimal place.)

6. Make a graph to reduce diameters to circumferences, and conversely, knowing that one circumference $= 3\frac{1}{7}$ diameters.

7. Construct graphical interest tables for 4 %, 5 %, 6 %, respectively.

Suggestion. Assume \$100 as principal. Then use 360 da. for known x, and \$4 for known y. Let 1 cm. = 60 da. on X axis; 1 cm. = \$1 on Y axis.

8. Construct graphs for reducing from dollars to English pounds; dollars to French francs; dollars to German marks, knowing that £1 = \$4.8665, 1 fr. = 19.3¢, and 1 M. = 23.8¢.

9. Construct a graph for uniform motion when distance 20 mi. is covered in $3\frac{1}{2}$ hr. Also graph when rate is given as so many miles an hour, as 14 mi. an hour. Find from first graph distance covered in $2\frac{3}{4}$ hr. Also time it takes to go 33 mi.

III. FUNCTIONALITY

123. A **constant** is a quantity that retains the same value throughout an investigation. Thus, in § 122, 1, the 2 and 3 are constants.

124. A **variable** is a quantity that changes in value. Thus, in § 122, 1, x and y are variables. The height of the mercury in a thermometer is a variable, as it is always slowly changing, either rising or falling. The speed of a train is a variable, always increasing or diminishing, though at times the change is very slight. The age of a man, the height of a tree, the cost of most articles, are all variables. Indeed, strictly speaking, nearly all quantities in nature are variables, though some of them change so slowly and slightly (as the length of an iron rod) that they are commonly regarded as constants.

Some variables change continuously while others change discontinuously. Thus, the age of a man changes continuously, while the price of an article, as the cost of butter, usually jumps from one integral number of cents to another. In the mathematical treatment such discontinuous variables are regarded as continuous.

Variables are commonly represented by the last letters of the alphabet, and constants by the others, though this rule is often broken.

125. Dependent and Independent Variables. Variables are usually connected, one depending for its value on the value of another.

For example, in a statistical graph which gives the temperatures throughout the hours of a day, we see that *the thermometer reading* depends on *the hour of the day.* In the formula $s = \frac{1}{2} gt^2$, as t increases s also increases as the body falls, both being continuous variables.

Similarly, we saw in the article on functions (§ 53) that any quantity containing x is a variable, if x is a variable.

Thus, $\frac{3x-2}{5}$ varies in value if x varies.

If $x=0$, $\frac{3x-2}{5} = -\frac{2}{5}$; if $x=1$, $\frac{3x-2}{5} = \frac{1}{5}$; if $x=2$, $\frac{3x-2}{5} = \frac{4}{5}$; if $x=3$, $\frac{3x-2}{5} = \frac{7}{5}$; etc.

The common plan is to let the letter y equal the function of x. Thus, in the preceding example, we would let $y = \frac{3x-2}{5}$.

Then, if $x=4$, $y=2$; if $x=5$, $y=2.6$; if $x=10$, $y=5.6$; etc.

It is customary to call x the *independent* variable and y the *dependent* variable.

126. The Use of Graphs to show Functional Relations. Such equations as $y = \frac{3x-1}{4}$ or $y = \frac{-3x-1}{4}$ can have graphs constructed for them as in § 116. Now by first locating a series of points and then drawing a *running line* through them, we see the graph showing a *continuous increase* in the value of x, and a corresponding *continuous increase* or *decrease* in the value of y.

1. Construct the graph for $y = 1 + 3x$, as in § 116.

Here we see that as x increases y also increases, and as x decreases y also decreases. Notice also that corresponding to *any* intermediate value of x between any two points there is a value of y which can be read off very approximately correct.

2. **Construct the graph for $y = 2 - 3x$.**

In this problem we see that as x *increases*, y *decreases* algebraically, and *vice versa.*

3. A small factory has a daily expense account of $15 for wages, materials, etc. The average sales per day per workman are $3. Make a graph from these conditions, using x for number of workmen, and $y =$ number of dollars of net profit.

4. A merchant's fixed charges for rent, interest on borrowed money, clerk hire, etc., are $4000 annually. His gross profit on sales averages 40 %. Construct a graph in which $x =$ number of thousands of dollars of annual gross sales, and $y =$ number of thousands of dollars of net profit.

From this graph find y when $x = 10, 20, 5$ respectively. We see here how net profits is a function of gross sales.

5. The formula for uniform motion is $s = vt$, in which $s =$ number of units of distance passed over, $v =$ number of units of distance passed over in one unit of time (velocity), and $t =$ number of units of time. For uniform motion v is constant.

The graph of § 122, 9 is of uniform motion, y taking the place of s and x that of t.

6. If the typesetting for a circular costs $3, and the paper and printing 6 ¢ a copy, construct graph from these data, letting $y =$ total cost, and $x =$ number of copies bought. From graph find cost of 500 copies. Find also number that can be bought for $8.

7. If the sundry charges for a lecture course (advertising, tickets, etc.) cost $100, and the talent costs $1.25 per ticket, construct graph showing net profit or loss for varying numbers of tickets disposed of, if tickets are sold at $1.50. Suppose 300, 500, 800 tickets are sold; what is gain or loss in each case?

CHAPTER V

SIMPLE EQUATIONS

I. DEFINITIONS

127. An **equation**, as already defined in § 54, is a statement that two quantities have the same numerical value.

128. Equations are of two kinds: equations of condition, and identities.

129. A **conditional equation** is one in which the two sides have the same numerical value only when a particular value, or values, are substituted for the unknown number.

Thus, $3x + 5 = 35$ is a conditional equation, since $x = 10$ is the only value of x which makes the two sides of the equation equal. Again, $x^2 - 5x + 6 = 0$ is "satisfied" only when $x = 2$, *or* when $x = 3$. If we put $x = 4$, the left side, or *member*, of the equation equals 2, reducing the equation to $2 = 0$, which is not true.

By an equation is commonly meant a conditional equation.

130. An **identity** is a statement that a quantity is equal to itself, or that two quantities are the same when both are reduced to their simplest form. The sign for an identity is $\equiv$.

For example, $ax + b \equiv ax + b$; $3 + 4 \equiv 2 + 5$;
$(x - 4)(x - 5) \equiv x^2 - 9x + 20$.

Identical equations, unlike conditional ones, hold true for *any* values of the letters contained in them. For instance, $x = 2, 3, 4, -2$, etc., substituted for x in the equation $(x - 4)(x - 5) \equiv x^2 - 9x + 20$ all satisfy it. Test this statement.

The given quantities in exercises in addition, subtraction, multiplication, division, and factoring, in Chapters I, II, III, set equal to the answers give rise to identical equations. The ordinary sign for equals, " =," is very often used in place of ≡. An identity is a declarative sentence. (See § 54.)

131. Conditional Equations Classified. Conditional equations are classified:

1. According to the degree (§§ 18, 90) of the unknown quantity, into: simple, quadratic, cubic, quartic, quintic, etc., equations. A **simple equation** is one of the *first degree;* a **quadratic equation** is one of the second degree; a **cubic** one of the third degree; etc. Notice the technical use of the word *simple.* It is easy to forget this definition, if one does not pay special attention to it.

2. According to the number of unknown quantities they contain, into: equations containing *one* unknown, and simultaneous equations.

132. Known and Unknown Quantities. The custom in algebra has been to denote unknown numbers by the last letters of the alphabet and known numbers by the others, but many authors are now using initials for the unknowns in problems. In science the rule is to use initials; as, v for velocity, p for pressure, etc.

133. The **root** of an equation is the value of the unknown number. Contrast root of an equation and root of a quantity (§ 13). To **solve** an equation is literally to untie the unknown number from the other quantities, thus finding its value.

134. To **verify a root** of an equation is to show that this value of the unknown number being substituted for the unknown in the given equation reduces it to an identity.

Before beginning the next article the pupil should carefully review §§ 54–58. The axioms should be thoroughly memorized.

II. EQUATIONS CONTAINING ONE UNKNOWN QUANTITY

135. Solution of Equations, especially fractional equations, by means of the axioms.

Solve and verify in the following:

1. $\frac{x-3}{2}-\frac{x-2}{3}=\frac{3x-21}{12}$.

SOLUTION. L. c. d. $= 12$.

$\therefore 6x-18-(4x-8)=3x-21$. (Mult. Ax. Equals, $\frac{x-3}{2}-\frac{x-2}{3}$ and $\frac{3x-21}{12}$, multiplied by the same number, 12, (the l. c. d.), give the equals, $6x-18-(4x-8)$ and $3x-21$.)

$\therefore 6x-18-4x+8=3x-21$. (Removing parentheses, § 36.)

$\therefore 6x-4x-3x=+18-8-21$. (Sub. Ax. The same quantity, $-18+8+3x$, subtracted from equals, $6x-18-4x+8$ and $3x-21$, leaves the equals, $6x-4x-3x$ and $+18-8-21$.)

$\therefore -x=-11$. (Adding on each side, § 31.)

$\therefore x=11$. *Ans.* (Div. Ax. Equals, $-x$ and -11, divided by the same quantity, -1, give the equals, x and 11.)

VERIFICATION. $\frac{11-3}{2}-\frac{11-2}{3}=\frac{3\times 11-21}{12}$; or, $4-3\equiv 1$.

2. $\frac{x-2}{5}-\frac{x-3}{4}=\frac{x-7}{10}$.

3. $\frac{x}{2}-2=\frac{x}{4}+\frac{x}{5}-1$.

4. $x+\frac{11-x}{3}=\frac{26-x}{2}$.

5. $\frac{3n}{5}+7-n=\frac{1}{2}$.

6. $\frac{x}{x+2}+\frac{4}{x+6}=1$.

7. $3+\frac{n}{.5}=10-\frac{n}{.2}$.

8. $\frac{x+4}{3x-8}-\frac{x+5}{3x-7}=0$.

9. $x=3x-\frac{1}{2}(4-x)+\frac{1}{8}$.

10. $\frac{7}{t}=\frac{23-t}{3t}+\frac{t-1}{4t}$.

11. $\frac{3n+7}{4n+8}=\frac{6n-2}{8n-5}$.

12. $\frac{8}{u}+\frac{u-3}{u+3}=\frac{u+1}{u-1}.$

13. $\frac{4}{x+2}+\frac{7}{x+3}=\frac{37}{x^2+5x+6}.$

14. $\frac{x}{7}=\frac{12x-60}{11}-\left(\frac{2x}{77}+1\right).$

15. $\frac{2x+1}{(x+2)^2}+\frac{2x+1}{x+2}=2.$

16. $\frac{1}{5-\frac{1}{x}}=\frac{2}{7}.$

17. $\frac{3}{4}-\frac{\frac{3}{4}x+\frac{3}{4}}{x+\frac{3}{4}}=\frac{\frac{3}{4}}{x+\frac{3}{4}}-\frac{3}{4}.$

18. $\frac{2x+19}{5x^2-5}-\frac{17}{x^2-1}=\frac{3}{1-x}.$

19. $\frac{1}{4}(4+\frac{3}{2}x)-\frac{1}{7}(2x-\frac{1}{3})=\frac{31}{28}.$

20. $3-\frac{1}{8}=\frac{1}{\frac{1}{3}+\frac{1}{t}}.$

21. $\frac{7y}{y+3}-\frac{5y}{1-y}=\frac{12(y^2-1)}{y^2+2y-3}.$

22. $\frac{1}{5}-\frac{3}{x-1}=\frac{2+\frac{x+4}{1-x}}{3}.$

23. $\frac{1}{3}(s-3)-\frac{1}{4}(s-8)+\frac{1}{5}(s-5)=0.$

24. $(x-1)^3+x^3+(x+1)^3=3x(x^2-1).$

25. $(x+\frac{5}{2})(x-\frac{3}{2})-(x+5)(x-3)+\frac{3}{4}=0.$

26. $(x+1)^2=\{6-(1-x)\}x-2.$ See § 21, 2.

27. $\frac{2z^2-z+3}{3z+2}-\frac{2z^2+3z-1}{3z-2}=\frac{3-6z-20z^2}{9z^2-4}.$

28. $\frac{x^2+3}{2(x^3-8)}+\frac{1}{6(x-2)}=\frac{2x-1}{3(x^2+2x+4)}.$

136. Special Processes of Solution. The regular method of solution is: (1) clear of fractions; (2) transpose; (3) collect; (4) divide through by the coefficient of x: but *any* method is permissible which solves or loosens x from the other quantities, provided the several steps can be justified. We can therefore use this rule: *Proceed in any manner that seems advantageous, by use of the axioms, to get x alone on one side, and only known quantities on the other side of the equation.*

Such irregular processes of solution often possess a distinct advantage over the normal process.

1. $\dfrac{x-1}{x-2}-\dfrac{x-5}{x-6}=\dfrac{x-3}{x-4}-\dfrac{x-7}{x-8}.$

Solution.

$$\frac{x-7}{x-8}-\frac{x-5}{x-6}=\frac{x-3}{x-4}-\frac{x-1}{x-2}. \qquad \text{(Sub. Ax.)}$$

$$\frac{x^2-13x+42-(x^2-13x+40)}{x^2-14x+48}=\frac{x^2-5x+6-(x^2-5x+4)}{x^2-6x+8}, \quad \text{(Adding.)}$$

or

$$\frac{2}{x^2-14x+48}=\frac{2}{x^2-6x+8}.$$

$$2x^2-12x+16=2x^2-28x+96. \qquad \text{(Mult. Ax.)}$$

$$16x=80. \qquad \text{(Sub. Ax.)}$$

$$x=5. \qquad \text{(Div. Ax.)}$$

Observe that transposing came first and clearing of fractions near the last instead of at first.

2. $\dfrac{1}{x-2}-\dfrac{1}{x-4}=\dfrac{1}{x-6}-\dfrac{1}{x-8}.$

Suggestion. Add on each side first.

3. $\dfrac{2}{x-14}-\dfrac{5}{x-13}=\dfrac{2}{x-9}-\dfrac{5}{x-11}.$

Suggestion. Transpose the two terms on either side of =, and then add in each member.

4. $\dfrac{9x+14}{12}-\dfrac{1}{7x}=\dfrac{5}{6x}-\dfrac{16-21x}{28}.$

Suggestion. Reduce fractions to mixed quantities first, and destroy $\frac{3}{4}x$.

5. $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}.$

Suggestion. Use Suggestions to 4 and 2.

6. $\dfrac{x^2-2x+1}{12}+\dfrac{5x-5}{24}=\dfrac{(x-1)^3}{12x}.$

Suggestion. Divide by $x-1$. Then, $x=1$ is a root, § 90.

If denominators contain both monomial and other expressions, it is often best to clear of monomial denominators *only* at first, and then simplify before proceeding to clear of the remaining denominators.

7. $\dfrac{9x+3}{27}+\dfrac{3x-6}{2x-5}=\dfrac{2}{3}+\dfrac{3x+22}{9}.$

Solution

$9x+3+\dfrac{81x-162}{2x-5}=18+9x+66.$ (Multiply through by 27, Mult. Ax.)

$\dfrac{81x-162}{2x-5}=81.$ (Sub. Ax.)

$\dfrac{x-2}{2x-5}=1.$ (Div. Ax. Dividing by 81.)

$x-2=2x-5.$ (Mult. Ax.)

$-x=-3,\ x=3.$ (Sub. and Mult. Ax.)

Let the student solve this problem by the usual method and compare the length with the solution just given.

8. $\dfrac{5x-3}{9}-\dfrac{10x+7}{18}+\dfrac{7x-29}{5x-12}=0.$

9. $\dfrac{48x+7}{24}-\dfrac{6x+1}{3}=\dfrac{x-16}{4(3x-57)}.$

10. $\dfrac{9(2x-3)}{14}+\dfrac{11x-1}{3x+1}=\dfrac{9x+11}{7}.$

11. $\dfrac{5z-6}{35}-\dfrac{z-5}{7z-112}-\dfrac{z}{7}=0.$

12. $\dfrac{10-x}{3}+\dfrac{13+x}{8}=\dfrac{7x-17}{x+21}-\dfrac{17+5x}{24}.$

137. Exercise in constructing Functions. Express the answers to the following questions in algebraic language. To test the correctness of the answers, assign some value or values to the letter or letters in both the given problem and the answer found, and see if the resulting numerical values agree. (See § 130.)

1. If a man earns $\$a$ a month and spends $\$x$ a quarter, how much will he save in a year? *Ans.* $12a-4x$ dollars.

Check. If he earns \$50 a month and spends \$100 a quarter, he saves \$200; also, $12a-4x=200$.

2. How far can a man go in 10 hours if he travels $x-1$ mi. an hour?

3. A has \$$c$ and B has \$$d$. If B gives A \$$n$, how much will each then have?

4. The width of a room is $m - 3$ ft. and its length $2 + m$ ft. more than its width. What is its perimeter, and what is its area?

5. In x lb. of gunpowder the niter composed 10 lb. more than $\frac{3}{4}$ of the total weight, the sulphur 3 lb. more than $\frac{1}{12}$ of it, and the charcoal the rest. How many pounds were there of each?

6. A man who had \$$d$ bought r lb. of rice at c¢ a pound, and s lb. of sugar at b¢ a pound. How many cents had he left?

7. A man sold a farm which had cost him \$$a$ to two parties, one taking b acres at \$$m$ an acre, and the other c acres at \$$n$ an acre. What was his gain per acre?

8. A fence is built across a rectangular field a rd. long and b rd. wide, dividing it into two parts of which one is a square. Find the number of acres in each part.

9. If a man can walk b mi. in x hr., how many feet does he walk in a second?

10. A man bought b yd. of cloth at \$$c$ a yard, and gained n% when he sold it. What did he sell it for?

11. What is the duty on n yd. of cloth worth \$$c$ a yard, taxed at r% ad valorem plus a specific duty of s¢ a yard.

12. The population of a town in 1900 was p and by 1910 it had fallen to q. What was the loss per cent?

13. A wheelman rode h hr. at m mi. an hour and then decreasing his speed he rode k hr. at n mi. an hour. He then returned in p hr. What was his rate back?

14. A cistern can be filled by one pipe in 20 min. and by a second pipe in 30 min. If the first pipe is open m min. and the second 4 min., what part of the cisternful is left to fill?

15. A passenger train running p mi. an hour and a freight train running f mi. an hour, being d mi. apart, approach each other, the passenger starting an hour after the freight. How far from the first train's starting point will they meet?

16. If n is an integer, express an even number; an odd number; three consecutive integers; three consecutive even integers; three consecutive odd integers. *Ans.* to 1st, $2n$; to 2d, $2n+1$.

17. Find the sum of ten consecutive integers of which the first is n. Apply your result to finding the sum of 237,130, 237,131, 237,132, $\cdots$, to ten numbers in a short way.

138. Exercise in expressing Sentences as Equations. Let $x =$ the principal unknown involved in the problems. Solve and verify in the equations after they are found. The sentence is a description of two quantities which are said to be equal.

1. If to 9 times a number 4 is added, the result is 1120.

SOLUTION. $9x \qquad\qquad + 4 \qquad\qquad = 1120,$

$\therefore x = 124.$

VERIFICATION. $9 \times 124 + 4 = 1120.$

2. The sum of two numbers, one 12 greater than the other, is 198.

3. Two numbers, the second of which is 3 more than twice the first, being added give 48.

4. The length of a rectangle is 7 ft. more than its width and its perimeter is 210 ft.

5. A man having completed $\frac{2}{5}$ of his journey finds that if he travels 30 mi. farther, only $\frac{3}{7}$ of the journey will remain.

6. In an election A received a majority of 65 votes over B and both received 943 votes.

7. Fifteen coins, dollars and quarter dollars, amount to $7.50.

8. A lawyer collects a debt for a client, takes 4% for his fee, and remits $207.60.

9. A fountain pen sold at $2.50 yields a profit of 85%.

10. A principal in 2 yr. 3 mo. 12 da. at 6% amounts to $1367.84.

11. A man walking 3 mi. an hour and then riding in an automobile 15 mi. an hour goes 63 miles in all, spending $\frac{2}{3}$ of the time walking. How long was he on the road?

12. The width of a field being $\frac{2}{3}$ of its length, when the width is increased by 5 ft. and the length by 10 ft., the area is increased by 400 sq. ft. What was its length?

13. How many hours would it take an express train going 48 mi. an hour and starting 3 hr. after an accomodation train going 30 mi. an hour to catch up with the latter?

14. A fruit dealer buying a certain number of oranges at 20¢ a dozen and twice as many at 25¢ a dozen sold them all at 35¢ a dozen, clearing $2.45. How many dozen did he buy at 20¢ a dozen?

15. An officer, attempting to arrange his men in a solid square, found that with a certain number of men on a side he had 34 men over, but with one man more on a side he needed 35 men to complete the square. How many men had he?

139. Exercise in translating Equations back into Sentences.

1. $3(x-2)=2(2x-13)$.

Ans. There is a certain number such that three times the difference between it and 2 is the same number as twice the difference between twice the number and 13.

2. $5x-4=3(x+4)$.

3. $\frac{2}{3}x-\frac{5}{6}=\frac{1}{4}x$.

4. $\frac{m-4}{4}=6-\frac{m}{3}$.

5. $\frac{7x+1}{5}-2x+\frac{4x+7}{7}=1$.

6. If B's money is $5 greater than A's, C's is $7 more than twice A's, and D's is $3 less than the sum of B's and C's, change the following equation into a sentence, making the conditions just given a part of the sentence:

$$x+(x+5)+(2x+7)+(3x+12-3)=168.$$

7. Change $y+(y+2)+(y+4)+(y+6)=96$ into a sentence.

III. PROBLEMS CONTAINING ONE UNKNOWN QUANTITY

140. Problems involving the Solution of Equations containing One Unknown Quantity.

The solution of *problems* consists of two parts:

(1) The translating of the problem into algebraic language, called the "statement" or "stating" of the problem.

(2) The solution of the equation.

Besides the solution there is always the verification. The verification of a *problem* is accomplished, not by substituting the answer in the equation found, but through testing it by the language of the problem, since the greatest danger of mistake here lies in the stating of the problem and not in the solution of the equation.

1. What number is that whose half, third, and 100 more equals twice the number and 2 more?

2. What number is that whose half, third, and fourth parts together equal 65?

Many of the types of problems now found in algebras have come down to us from remote antiquity. Some of them were given in arithmetics, and were called puzzles or enigmas because they were quite difficult to solve by arithmetic. Some authors called them amusing, or joke problems. The purpose in giving them was to sharpen the wits, to offer recreation, and to while away the time. We give a short list of such problems.

3. A tower standing for $\frac{1}{4}$ of its length in the earth and $\frac{1}{5}$ in the water rises 100 feet in the air. What is its total height?

(From Bamberg Arithmetic, the first printed arithmetic, 1482.)

4. A vinedresser hires a laborer with the understanding that for every day he worked he should receive 10 M., and for every day he was idle he should forfeit 12 M. After 40 da. the laborer quit, receiving no money. How many days did he work?

(Bamberg Arithmetic.)

5. Heap (the unknown), its $\frac{2}{3}$, its $\frac{1}{2}$, its $\frac{1}{7}$, its whole, it makes 33. What is the number? (Ahmes papyrus, 1700 ? B.C.)

6. A barrel has three faucets, by the first of which it is emptied in 2 days, by the second in 3 days, and by the third in 4 days. In what time will the barrel be emptied if all are left open? (Bamberg Arithmetic.)

7. A hound pursues a hare which is 100 leaps ahead of the hound. If the hare makes 12 leaps while the hound makes 15, how many leaps will the hare make before the hound catches her, if the hound's leaps are the same length as those of the hare's? (Bamberg Arithmetic.)

8. At what time between 3 and 4 o'clock are the hands of a clock (1) together; (2) at right angles; (3) opposite.

Suggestion. — In solving clock problems use minute spaces as the unit of measure, and note that in any specified time whatever the minute hand traverses exactly 12 times as many spaces as the hour hand.

9. In an old Chinese arithmetic whose origin antedates 1250 B.C., there is given this problem: In the middle of a square pond 12 units on a side there grows a reed which rises 1 unit above the water. If the reed be pulled over to the middle point of one of the edges, it just reaches the top of the water. What is the depth of the water in the center?

10. A thief who stole a sum of gold was obliged to pass three doorkeepers on his way out. The first of these demanded the half of the stolen amount, but gave back to the thief 100 florins. The second demanded the half of the remainder, and then gave back 50 florins. The third demanded half of the last remaining sum, and gave back 25 florins to the thief, who now found he had left only one third of the sum stolen. How many florins did the thief steal? (Jacob Koebel, 1470–1533 A.D.)

11. A merchant adds annually $\frac{1}{3}$ of his money diminished by £100 which he spends each year, and after three years is twice as rich as at first. Find the amount of his money at first. (From Newton's Universal Arithmetic, *i.e.* Algebra, 1707.)

12. A scheme for one person to tell to another the result of a series of operations performed on a number secretly selected by the latter. Example. Denote by x the number secretly selected. Tell the person to take *any* number he wishes, and multiply it by 6, say, add 4 to the result, then add 15, then subtract 1, then divide the remainder by 3, then subtract twice the number he started with, then to square the result, subtract 6 from it, and divide the result by 5. Now tell him he has 6 for answer. Explain how you know by algebra.

13. Two parallel chords equally distant from the center of a circle intercept arcs whose sum is $\frac{2}{15}$ of the remaining parts of the circumference. Find the number of degrees in each part of the circumference.

14. Two lines intersect so that $2n - 1$ degrees represents the size of one of the angles of the two pairs of vertical angles and $3n + 6$ the number of degrees in one of the angles of the other pair. Find the number of degrees in each of the four angles.

15. If two lines are cut by a transversal so that $2n - 5$ is the number of degrees in one of the interior angles and $10n - 50$ the sum of all four interior angles, find the number of degrees in each of the eight angles formed by the transversal with the two parallel lines.

16. The perimeter of a parallelogram is 360 rd. and the ratio of two adjacent sides is $2:5$. How many rods are there in each side?

17. Two adjacent angles of a parallelogram are $2n + 10$ and $3n - 20$ degrees. How many degrees are there in each angle?

18. An inscribed angle is subtended by $2n - 10$ degrees and the rest of the circumference is denoted by $5n + 20$ degrees. How many degrees are there in the angle?

19. The arcs intercepted between two vertical angles formed by two chords which intersect within a circle are denoted by $3y - 16$ degrees and $2y + 7$ degrees. The remaining arcs of the circumference together equal $7y - 15$ degrees. How many degrees are there in the angle between the chords?

IV. SIMULTANEOUS EQUATIONS

141. Simultaneous Equations are sets of equations whose unknown quantities have the same values in the different equations. Thus in $\begin{cases} 2x + 3y = 19, \\ 13x - 2y = 16, \end{cases}$ x has the value 2 and y the value 5 in both equations. Simultaneous equations are solved by a process called elimination.

142. Elimination is the process of deriving from two or more equations containing two or more unknown quantities a single equation containing one unknown quantity. The two most important methods of elimination are:

(1) By Substitution. (2) By Addition and Subtraction.

143. Elimination by Substitution is accomplished by finding the value of one unknown from one of the given equations and *substituting* this value for the same unknown in the other equation.

a. Since two equations are being handled, in order to refer to them conveniently, they are numbered (1) and (2). Changed forms of these equations will be marked (1_1), (2_1), (1_2), etc.

1. Given $\begin{cases} (1)\ \dfrac{7}{x-3} = \dfrac{8}{y-5}, \\ (2)\ \dfrac{9}{2x-1} = \dfrac{5}{3y+4}, \end{cases}$ to find values of x and y which satisfy both equations.

SOLUTION. (1_1) $7y - 35 = 8x - 24$ (Mult. Ax.)

(1_2) $7y = 8x + 11$ (Sub. Ax.)

(1_3) $y = \dfrac{8x + 11}{7}$ (Div. Ax.)

(2_1) $27y + 36 = 10x - 5$ (Mult. Ax.)

$27\left(\dfrac{8x + 11}{7}\right) + 36 = 10x - 5$ (§ 143.)

$216x + 297 + 252 = 70x - 35$ (§ 105, Mult. Ax.)

$146x = -584$ (Sub. Ax.)

$x = -4$. *Ans.* (Div. Ax.)

(1_3) $y = \dfrac{8 \times -4 + 11}{7} = -3$. *Ans.* (Substituting -4 for x.)

VERIFICATION (1) $\dfrac{7}{-4-3} \equiv \dfrac{8}{-3-5}$; (2) $\dfrac{9}{-8-1} \equiv \dfrac{5}{-9+4}$.

2. $\begin{cases} (1)\ 9x + 8y = 57, \\ (2)\ 6x + 7y = 48. \end{cases}$

3. $\begin{cases} (1)\ x - \dfrac{4y - 9}{11} = 5, \\ (2)\ \dfrac{9}{2} - \dfrac{x + 5}{3} = -3y. \end{cases}$

144. Elimination by Addition and Subtraction.

1. Given $\begin{cases} (1)\ \dfrac{x}{4} + \dfrac{y}{5} = 5, \\ (2)\ \dfrac{2x}{3} + y = 18, \end{cases}$ to find values of x and y which satisfy both equations.

SOLUTION. $(1_1)\ \ 5x + 4y = 100$ $(2_1)\ 2x + 3y = 54$ (Mult. Ax.)

$(1_2)\ \ 10x + 8y = 200$ (Multiplying (1_1) by 2, Mult Ax.)

$(2_2)\ \ \underline{10x + 15y = 270}$ (Multiplying (2_1) by 5, Mult. Ax.)

$7y = 70$ (Subtracting (1_2) from (2_2), Sub. Ax.)

$y = 10$. *Ans.* (Div. Ax.)

$(2_1)\ \ 2x + 3 \times 10 = 54.$ (Substituting its value for y.)

$x = 12$. *Ans.*

VERIFICATION (1) $\dfrac{12}{4} + \dfrac{10}{5} \equiv 5$; (2) $\dfrac{2 \times 12}{3} + 10 \equiv 18$.

2. **Rule.** *After reducing both equations to the form $ax + by = c$, each equation is multiplied through by such a number that the coefficients of x (or y if easier) are made the same in both new equations. Then, one of these equations is subtracted from the other (or added when the like coefficients have opposite signs).*

3. $\begin{cases} (1)\ 2x + 3y = 18, \\ (2)\ 3x - 2y = 1. \end{cases}$

4. $\begin{cases} (1)\ \dfrac{4x + 5y}{40} = x - y, \\ (2)\ \dfrac{2x - y}{3} - 2y = \dfrac{1}{2}. \end{cases}$

145. Elimination by Comparison is less important than the other methods. It is performed by finding the value of the same unknown from both equations and setting these values equal to each other.

146. General Exercise in Elimination with Two Unknowns. Solve each problem by both substitution and addition and subtraction and compare the methods.

1. $\begin{cases} 2x + 7y = 38, \\ 3x + 4y = 31. \end{cases}$

2. $\begin{cases} 17a - 18b = 52, \\ 5a - 12b = 22. \end{cases}$

3. $\begin{cases} 29x + 85y = 31, \\ 13x - 43y = 95. \end{cases}$

4. $\begin{cases} .5x + y = 2.75, \\ 3.4x + .02y = 1.75. \end{cases}$

5. $\begin{cases} 5h + 2k = -7, \\ \dfrac{2h + k + 6}{h - 2k - 3} = -\dfrac{2}{7}. \end{cases}$

6. $\begin{cases} \dfrac{a + 2b}{4} + \dfrac{2a + b}{3} = 1, \\ 8a + 7b = 12. \end{cases}$

7. $\begin{cases} \dfrac{4}{m-2} + \dfrac{3}{n+1} = 1, \\ \dfrac{1}{m-2} + \dfrac{6}{n+1} = \dfrac{9}{8}. \end{cases}$

8. $\begin{cases} \dfrac{3x - 5y}{2} + 3 = \dfrac{2x + y}{5}, \\ 8 - \dfrac{x - 2y}{4} = \dfrac{x}{2} + \dfrac{y}{3}. \end{cases}$

9. $\begin{cases} \dfrac{6}{x} + \dfrac{12}{y} = -1, \\ \dfrac{8}{x} - \dfrac{9}{y} = 7. \end{cases}$

SUGGESTION. Solve without clearing of fractions for $\frac{1}{x}$ and $\frac{1}{y}$, since clearing introduces term xy.

147. Special Methods of Elimination. Instead of eliminating immediately, it is often better to multiply each equation by such a multiplier that when the two resulting equations are added or subtracted a new equation results, which, though still containing both unknowns, has small coefficients. This new equation can now be combined with one of the old equations, or with another new equation obtained in the same way. Any pair of such equations are said to constitute a *system*, since they suffice to find the values of the unknowns.

1. Given $\begin{cases} (1)\ 13x - 29y = 97, \\ (2)\ 17x - 39y = 129, \end{cases}$ to find values of x and y.

SOLUTION.

$$\begin{array}{lrcll} (1) & 52x - 116y &=& 388 & \text{(Mult. Ax.)} \\ (2) & \underline{51x - 117y} &=& \underline{387} & \text{(Mult. Ax.)} \\ (3) & x + \quad y &=& 1 & \text{(Sub. Ax.)} \\ (3_1) & 13x + \ 13y &=& 13 & \text{(Ax. ?)} \\ (1) & \underline{13x - \ 29y} &=& \underline{97} & \\ & 42y &=& -84 & \text{(Ax. ?)} \\ & y &=& -2. \quad \textit{Ans.} & \text{(Ax. ?)} \end{array}$$

(3) $x + (-2) = 1$; $x = 3$. *Ans.*

Verification. (1) $13 \times 3 - 29 \times -2 \equiv 97$; (2) $17 \times 3 - 39 \times -2 \equiv 129$.

(*a*) If the student will solve this problem by the regular process of solution, that is by multiplying (1) through by 17 and (2) through by 13, he will find that the labor of solution is two or three times as great as that of the preceding solution. Evidently the plan is to multiply both members of each of the two equations by two small numbers so as to bring the corresponding coefficients *close together* in value instead of to make them equal.

2. $11x - 21y = 26$; $21x - 40y = 50$.

3. $19x + 35y = 127$; $28x + 53y = 190$.

4. $23x + 29y = -1$; $29x + 23y = 53$.

Suggestion. Add and divide through by 52, getting (3); then subtract and divide through by 6, getting (4). Use (3) and (4) to finish solution.

148. Systems of Equations. Equivalence. Two systems of equations (§ 147) are **equivalent** when one set can be derived from the other by axiomatic processes, and the values of the unknowns obtained from each set are the same. Thus, we saw in the last article that values of the unknowns derived from using a new third equation along with one of the given equations verified in both of the given equations; also in Ex. 4, that values of the unknowns obtained from the two *new* equations (3) and (4) verified in the original equations.

It may be said here that, in general, throughout simple equations, values of the unknowns found from a set of derived equations will verify in the original ones. Nevertheless the student should make it a rule to test all answers by substituting them in the original equations.

149. Simultaneous Equations Containing Three or more Unknowns.

1. Given $\begin{cases} (1)\ 2x + 3y - z = 21, \\ (2)\ 6x - 7y + 5z = 55, \\ (3)\ 9x + 5y - 2z = 71, \end{cases}$ to find values of x, y, z which will satisfy each of these equations.

Solution. Of the unknowns z is most easily eliminated.

$$\begin{array}{llr}
(1_1) & 10x + 15y - 5z = 105 & \text{(Mult. Ax.)} \\
(2) & \underline{6x - 7y + 5z = 55} & \\
(4) & 16x + 8y = 160 & \text{(Ax.?)} \\
(4_1) & 2x + y = 20 & \text{(Ax.?)}
\end{array}$$

$$\begin{array}{lrcll}
(3) & 9x + 5y - 2z & = & 71 & \\
(1_2) & \underline{4x + 6y - 2z} & = & \underline{42} & \text{(Ax.?)} \\
(5) & 5x - y & = & 29 & \text{(Ax.?)} \\
(4_1) & \underline{2x + y} & = & \underline{20} & \\
 & 7x & = & 49 & \text{(Ax.?)} \\
 & x & = & 7. & \textit{Ans.}\ \text{(Ax.?)} \\
(4_1) & 2 \times 7 + y & = & 20. & \text{(Substituting its value 7 for } x\text{).} \\
 & y & = & 6. & \textit{Ans.}
\end{array}$$

$$(1) \qquad 2 \times 7 + 3 \times 6 - z = 21. \quad \text{(Substituting their values for } x \text{ and } y\text{.)}$$

$$z = 11. \quad \textit{Ans.}$$

Verification (2) $6 \times 7 - 7 \times 6 + 5 \times 11 \equiv 55$,

(3) $9 \times 7 + 5 \times 6 - 2 \times 11 \equiv 71$.

2. $\begin{cases} 4x - 3y + z = 9, \\ 9x + y - 5z = 16, \\ x - 4y + 3z = 2. \end{cases}$

3. $\begin{cases} 12l + 5m - 4n = 29, \\ 13l - 2m + 5n = 58, \\ 17l - m - n = 15. \end{cases}$

4. Make a rule for solving sets of three simultaneous equations.

5. $\begin{cases} u + v + 6t = 63, \\ v + t - 6u = -130, \\ 3u + v - t = 76. \end{cases}$

6. $\begin{cases} \frac{1}{6}x - \frac{1}{5}y + \frac{1}{4}z = 3, \\ \frac{1}{5}x - \frac{1}{4}y + \frac{1}{5}z = 1, \\ \frac{1}{4}x - \frac{1}{3}y + \frac{1}{2}z = 5. \end{cases}$

7. $\begin{cases} 2p - 3q = 3, \\ 3q - 4r = 7, \\ 4r - 5p = 2. \end{cases}$

SUGGESTION. Eliminate q, using (1) and (2), getting (4). Then take (3) and (4) together.

8. $\begin{cases} \dfrac{2}{x} + \dfrac{1}{y} - \dfrac{3}{z} = 0, \\ \dfrac{3}{z} - \dfrac{2}{y} - 2 = 0, \\ \dfrac{1}{x} + \dfrac{1}{z} - \dfrac{4}{3} = 0. \end{cases}$

SUGGESTION. Do not clear of the denominators, but solve directly for $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, and then get x, y, z. Solve as in Ex. 7.

9. $\begin{cases} 3x + 6y + 2z + u = 2, \\ x - y - 3z - 4u = 3, \\ x + 2y - 2z - 2u = 0, \\ 2x + y - z - 3u = 5. \end{cases}$

SUGGESTION. Eliminate x first. Combine (2) and (3), eliminating x, getting (5). Then combine (1) and (2), eliminating x, getting (6). Then combine (2) and (4), eliminating x, getting (7). Now solve (5), (6), and (7) as in preceding problems.

150. Problems involving the Solution of Equations containing One or More Unknowns. The choice of the number of unknowns is left to the student. The same problem can often be solved equally well by using different numbers of unknowns.

1. A gentleman divides 2 dollars among 12 children, giving to some 18¢ each, and to the others 14¢ each. How many were of each class?

2. A grocer has two sorts of sugar, one worth 5¢ and the other $6\frac{1}{2}$¢ a pound. How many pounds of each sort must be taken to make a mixture of 100 lb. worth $5.60?

3. Two sides of an equiangular triangle are $15x-6y$ and $20x-9y$, and the third side is 15. Find x and y.

4. How many gallons each of cream containing 30% fat and milk containing 5% fat shall be mixed so as to produce 10 gal. of mixture containing 25% of fat?

5. If 6, 8, and 11 ft. respectively are the sides of a triangle, find the segments into which the bisector of the angle between 6 and 8 sides divides the third side. What are the segments when an exterior angle is bisected? (See § 51.)

6. In an athletic meet the team from town A won the championship with a total of 36 points, taking five firsts, three seconds, and two thirds. The team from town B got four firsts, two seconds, and four thirds, with a total of 30 points, and the team from town C got two firsts, six seconds, and five thirds, and a total of 33 points. How many points did a first, second, and third respectively count?

7. Earth and Venus move around the sun in orbits approximately circular, one moving around the sun in a shorter period than the other so that at one time they and the sun are in line with Venus between the other two, and at another time they are in line in a different place with the sun between the other two. When nearest together Earth and Venus are only 25.6 millions of miles from each other; when farthest apart they are 160 millions of miles distant from each other. Find the distance of each planet from the sun.

V. GRAPHICAL SOLUTION OF SIMULTANEOUS EQUATIONS

151. Value of the Graphical Method. Whether the algebraic or the graphical method will give the answer more quickly depends on the problem and the skill of the user, but the algebraic solution is always accurate, while the graphic solution is often only approximate. One of these methods can be used to check the answer obtained by the other, though verification is also available for both. But for the student now the main value of the graphical solutions consists in the fact that they throw a strong light on the nature of simultaneous equations and on peculiarities that may occur in them.

152. Graphical Solution of Pairs of Equations containing Two Unknowns. 1. Solve (1) $\frac{x}{3}-\frac{y}{6}=\frac{1}{2}$, (2) $\frac{x}{5}-\frac{3y}{10}=\frac{1}{2}$, by the graphical method, and check with an algebraic solution.

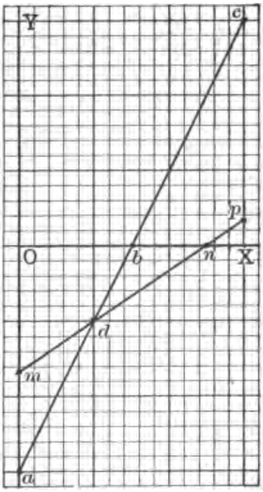
Graph Solution.

(1_1) $2x - y = 3$

(x, y)	Pt.
$(0, -3)$	a
$(1.5, 0)$	b
$(3, 3)$	c

(2_1) $2x - 3y = 5$

(x, y)	Pt.
$(0 - 1.67)$	m
$(2.5, 0)$	n
$(3, .33)$	p



The coördinates that satisfy *both* equations are those of the point d, *where the graphs cross.* The coördinates of d are $x = 1$, $y = -1$. *Answers.*

Algebraic Solution.

$$\begin{array}{lrl} (1_1) & 2x - y = & 3 \\ (2_1) & 2x - 3y = & 5 \\ \hline & 2y = & -2 \\ & y = & -1. \ \textit{Ans.} \end{array} \qquad \begin{array}{ll} (1_1) & 2x - (-1) = 3. \ (\S\ 144.) \\ & x = 1. \ \textit{Ans.} \end{array}$$

2. $\begin{cases} 3x + 4y = 10, \\ 4x + y = 9. \end{cases}$

3. $\begin{cases} 5x + 6y = 17, \\ 6x + 5y = 16. \end{cases}$

4. $\begin{cases} 2x + 3y = 43, \\ 10x - y = 7. \end{cases}$

5. $\begin{cases} 11x - 14y = 14, \\ 5x + 7y = 41. \end{cases}$

6. $\begin{cases} 18x - 10y = 29, \\ 14x - 15y = 24. \end{cases}$

7. $\begin{cases} 7x - 9y = -22, \\ x = -4. \end{cases}$

8. $\begin{cases} 11d + 16t = 64, \\ 7d - 12t = 13. \end{cases}$

9. $\begin{cases} 8x = 5y, \\ 13x = 8y + 1. \end{cases}$

10. $\begin{cases} 11x - 3y = 321, \\ 5x - 16y = 585. \end{cases}$

11. $\begin{cases} 7x + 2y = 76, \\ 2x - 3y = 11. \end{cases}$

Suggestion to Ex. 10. Since the numbers are large, the centimeter can not be used as the unit of measure. The 2-millimeter unit can be employed in this case.

153. Consistent Equations. The student may ask himself the question whether if three equations, each containing two unknowns, were taken at random, the values of x and y obtained by solving the system formed out of the first two equations would have the same values as those obtained by taking the system formed from the second and third equations, or by taking the system formed from the first and third equations.

On the other hand, if the third equation was obtained by combining the first two in some way, would the values of x and y found by solving the system formed from the first two satisfy the third equation?

Example.

$$\begin{array}{lrl} (1) & 11x - 10y = & 14 \\ (2_1) & 10x + 14y = & 82 \\ \hline (3) & x - 24y = & -68 \end{array} \qquad (2)\ 5x + 7y = 41$$

Construct now the graphs of equations (1) and (2) and (3) to the same axes. What do you learn?

Test problems of § 147 in the same way.

154. Inconsistent Equations. Construct the graphs for the following equations on the same axes:

(1) $4x-2y=7$. (2) $4x-9y=-21$. (3) $3x-4y=-8$.

Solve the system formed from (1) and (2) algebraically and compare the answers with those on diagram obtained graphically. Do the same with the system (1) and (3), and with the system (2) and (3).

Write sets of three equations at random and see whether their graphs intersect in three points or one point.

155. Systems of Equations in which One of the Two Given Equations can be derived from the Other.

Construct the graphs for the system

$$(1)\ 2x-5y=11;\ (2)\ \frac{8x-44}{5}=4y,$$

in which (2) can be derived from (1) by first multiplying through by 4, then transposing, and then dividing through by 5.

What do you learn about these graphs?

Test your answer by making and testing other similarly constructed systems. Is then the system of marking equations, in which the equation marked (1) at the start is marked (1) with subscripts throughout, justified?

156. Systems of Equations which differ only in their Known Terms.

Construct the graphs on the same axes for the system

$$(1)\ 2x-5y=12,\ (2)\ 4x=10y+6,$$

in which, obviously, (2) can be changed into $2x-5y=3$.

What do you find true of these graphs?

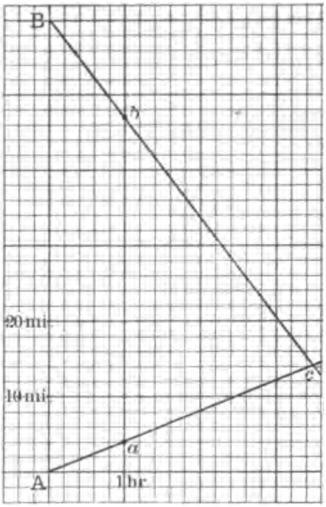
Test your answer by constructing similar systems which differ only in their known terms.

We learn then that whenever by transposing, and multiplying or dividing through, two equations can be made to differ in their known terms only, *their graphs are parallel.*

a. The graphs of three equations each having two unknowns may intersect in 0, or 1, or 2, or 3 points (not at infinity), or in every point. Explain.

157. Solution of Problems by Graphs. — Verify the results in Exs. 1–5, 8, 9, following, with algebraic solutions.

1. Two men set out from two towns *A* and *B*, 60 mi. apart, the one from *A* walking 4 mi. an hour, and the one from *B* riding in an automobile at the rate of 13 mi. an hour. How far from *A* will they meet, and in how many hours from the time they start?



Solution. Take a vertical axis 6 cm. long to denote 60 mi., marking its extremities *A* and *B*, *A* below and *B* above. To the right of *A* let the centimeters denote hours. Then thinking of *A* as origin, locate point *a* as (1, 4) in which 1 is one hour and 4 is 4 of the 60 mi. in the line *AB*. Again, thinking of *B* as origin, locate point *b* as 1 hr. to the right of *B* and 13 of the 60 mi. units below *B*. Now draw *Aa* and *Bb* and extend them until they meet at *c*. The number of units from *c* down to the horizontal line through *A* gives the number of miles from *A* to where they meet. Also the abscissa of *c* gives the time when they meet.

Notice that $y = 4x$ is the equation of *Aa*, and $y = -13x$ that of *Bb* as in § 122, but different unit lengths are used here for x and y.

Verification by an Algebraical Solution. Let $x =$ number of miles from *A* to where they meet. Then $\frac{x}{4} = \frac{(60 - x)}{13}$, whence $x = 14\frac{2}{17}$ mi. This shows that the squared paper solution is not quite accurate, but gives a good approximation to the answer.

2. Two trains start from New York and Washington at the same time, 8 A.M., the one from New York running 45 mi. an hour and the other 40 mi. an hour, and Washington and New York are 228 mi. apart. How far from New York will they meet, and at what hour of the day?

3. Two steamers leave New York and Queenstown, Ireland, the same day, Monday, the first at 3 P.M. and the other at 6 A.M.,

both Greenwich time. The first makes 600 and the second 500 geographical miles a day, and the two places are 2780 geographical miles apart. How far from Queenstown will they meet, and at what time?

SUGGESTION. Let the vertical line AB denote 2780 mi., A being New York. Then start the graph for the New York steamer 9 hr. to the right of A as origin, point A being at 6 A.M.

4. A man made a trip to the country with an automobile, running 18 mi. an hour. The machine broke down, and he had to return on a bicycle, making 10 mi. an hour. He was away from home $3\frac{1}{2}$ hours in all; how far did he go?

SUGGESTION. Let MN represent a horizontal line which stands for $3\frac{1}{2}$ hr. Draw the graphs for the motion from the points M and N.

5. The 20th Century Limited leaves New York (1911 schedule) at 4 P.M., traveling 53 mi. an hour, following the Day Express which leaves New York at 9.40 A.M., traveling 34 mi. an hour. How far from New York and at what time will the fast train catch the other?

SUGGESTION. Let MN represent the difference in time of starting. Draw both graphs northeast from M and N.

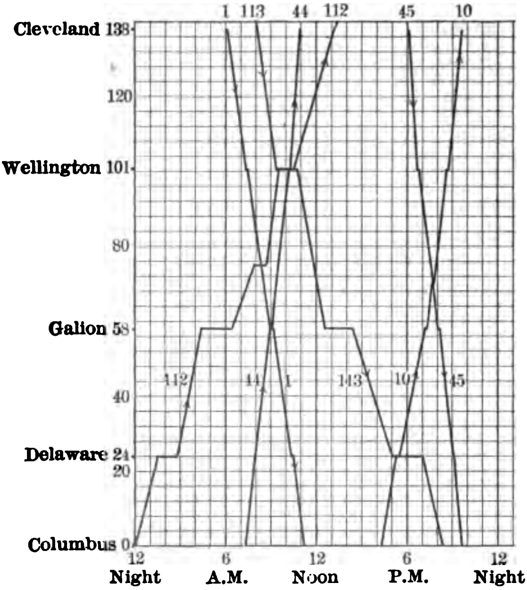
6. Construct a graph for the movement of a suburban train which runs on the following schedule:

STA.	MI.	ARRIVE	DEPART
A	0		6 A.M.
B	7	6.15	6.18
C	12	6.25	6.27
D	18	6.35	6.40
E	25	6.50	

SUGGESTION. Notice when the train is standing still the graph runs horizontally, so that the graph for a train stopping and starting is zigzag.

7. We give next a "*train chart*" used by train dispatchers on railroads to make time schedules. On a large chart lines are drawn (or threads are stretched, held in position by pins), as in

accompanying diagram, which show where and when trains start, stop, and pass each other. The stops of the four passenger trains (Nos. 1, 10, 44, 45) are so short that they hardly show. The freight (Nos. 112, 113) stops are much longer. Any number of trains can be put on the same diagram.



8. A can do a job in 10 days and B the same in 13 days. In what time can both do it?

Suggestion. Let any convenient vertical length *AB* denote the work, *i.e.* one times the work. Mark the days to the right from both *A* and *B*. Let *a* be 10 days directly east from *B* and *b* 13 days east of *A*. Join *A* and *a* and *B* and *b*. Where these graphs cross is the answer sought.

9. Three faucets fill a cistern, the first in 20 min., the second in 30 min., and the third in 40 min. In what time will all fill it?

Suggestion. First find the time for the first two, and then for this result and the third.

VI. LITERAL EQUATIONS

158. Solution of Applied Formulas for any Letter. The following formulas come from arithmetic, geometry, physics (including sound, light, heat, electricity), shop-work, trigonometry, engineering, etc. They are all solved by use of the axioms in much the same manner as the equations of § 135.

Solve the following formulas for each letter in them (excepting known numbers like π) not given directly:

1. $s = ba$ (area of a rectangle = base × altitude.)

Solution. $ba = s$. (Reversing members.)

$a = \frac{s}{b}$. *Ans.* (Div. Ax.). $b = \frac{s}{a}$. *Ans.* (Div. Ax.)

2. $c = \frac{b}{h}$. 3. $s = \frac{a}{h}$. 4. $t = \frac{a}{b}$.

5. $d = \frac{w}{v}$. 6. $w = fd$. 7. $c = pn$.

8. $a = \frac{1}{2}bh$. 9. $A = \pi ab$. 10. $ST = st$.

11. $t = \frac{1}{k}$. 12. $F = \frac{Wa}{g}$. 13. $l = \frac{\pi dn}{360}$.

14. $\frac{PV}{T} = \frac{P'V'}{T'}$. 15. $E = \frac{nCN}{10^8}$. 16. $R = 3 + \frac{V}{6}$.

17. $P \times P_a \times S_a = W \times W_a \times s_a$. 18. $b = 1.155(f - 0.2)$

19. $R = 0.3788\,GT$. 20. $\frac{P}{W} = \frac{dr}{2\pi lR}$. 21. $a = b(1 + r)$.

22. $a = p(1 + rt)$. 23. $v = 4N(l + r)$. 24. $Mts = M't's'$.

25. $d = d' + \frac{2}{p}$. 26. $A = \frac{1}{2}a(b + b')$. 27. $H = \frac{PLAN}{33000}$.

28. $C = \frac{SE}{Sb + R}$.

Solution for S. $SCb + CR = SE$. (Mult. Ax.)

$SCb - SE = -CR$. (Sub. Ax. Unknowns to left, knowns to right member.)

$S(E - Cb) = CR$. (Changing signs, Mult. Ax., and factoring.)

$S = \frac{CR}{E - Cb}$. *Ans.* (Div. Ax.)

Other answers:

$$E = \frac{SbC + RC}{S};\ R = \frac{SE - SbC}{C};\ b = \frac{SE - CR}{CS}.$$

29. $C = \frac{EP}{b + RP}$. **30.** $c = \frac{SEP}{Sb + RP}$. **31.** $R = \frac{gs}{g + s}$.

32. $s = \frac{lr - a}{r - 1}$. **33.** $S = \frac{W}{W - W'}$. **34.** $\frac{P}{W} = \frac{p}{2\pi R}$.

35. $\frac{1}{F} = \frac{1}{D} + \frac{1}{D'}$. **36.** $S = \frac{w_1}{w_1 + w_2 - w_3}$. **37.** $\frac{2R - h}{h} = \frac{l}{2p}$.

38. $\frac{P + p}{2p} = \frac{P}{P'}$. **39.** $l = \frac{\pi D + \pi D'}{2} + 2d$. **40.** $a = \frac{2n - 4}{n}$.

41. $\frac{1}{R} = \frac{1}{r} + \frac{1}{r'} + \frac{1}{r''}$. **42.** $ms(t - t_m) = m's'(t_m - t')$.

Pupils should examine geometries, books on physics, and shop mathematics for similar formulas, and solve those found for any letter. But if in any formula the unknown appears to the second or higher powers, the method of solution is different from that here explained. It will be given later.

159. Making and Using of Formulas in Arithmetic.

1. If b represents base, r rate, and p percentage, make a formula giving the value of p. Solve this formula for b; for r.

a. The word rate means *ratio*, that is, the ratio of the percentage to the base; or, if r = rate, $r = \frac{p}{b}$. In the solution of problems it is often preferable to let $r\%$, or $\frac{r}{100}$ = the rate, thus making r ordinarily an integral or mixed number, rather than a decimal. Then $\frac{r}{100} = \frac{p}{b}$.

Solve the following numerical problems by substituting in the appropriate formula:

2. A trader having \$1960 spent 15 % of it. What sum did he spend?

3. If I sell $\frac{5}{9}$ of a carload of wheat for what $\frac{2}{3}$ of the whole carload cost, what is the gain per cent?

SUGGESTION. Find first what part of cost whole carload would be sold for, and then find gain.

4. $68\frac{3}{4}$ is $9\frac{1}{6}$ % of what number?

5. If b represents base, r rate, a amount, or sum of base and percentage, and d "difference," or base less percentage, write a formula for value of a; also a formula for value of d. Solve the latter for b.

6. A has \$3009, which is 18 % more than B has. How much has B? Also A has 15 % less than C. How much has C?

7. If p represents principal, r rate, t number of years principal bore simple interest, and a amount, make a formula giving the value of a.

a. This formula can be used to solve *any* problem in simple interest.

8. Find the amount of \$520 at 5 % for 4 yr. 4 mo. 24 da. by using formula.

9. Find the rate when \$1652.64 is amount, principal is \$1320, and time is 3 yr. 7 mo. 6 da. Solve the formula for r first.

10. Find the principal when the amount is \$279.18, rate is 3 %, and time is 1 yr. 1 mo. 6 da. Solve the formula for p first.

11. Find the time when the amount is \$1148, the rate is $4\frac{1}{2}$ %, and the principal is \$1025. Solve the formula for t first.

12. Potatoes whose value was p dollars were shipped to a commission merchant to sell, who charged r % for selling or buying. With proceeds from the sale of the potatoes he was instructed to buy salt. What is the value of the salt he should ship back? Suppose $p = 3198$ and $r = 2\frac{1}{2}$.

13. If goods that cost \$$c$ are sold for \$$s$, what is the rate per cent r of gain? Find rate r when $c = 450$, $s = 531$.

14. Goods that cost c¢ were marked to sell at m¢, but r' % was thrown off from the marked price. What was the actual gain per cent r?

Given $c = 8$, $m = 12$, $r' = 16\frac{2}{3}$, to find r.

15. Three discounts of r %, s %, t % were taken from a bill of \$$b$. What was the net cost c?

Given $b = 2250$, $r = 20$, $s = 15$, $t = 8$, to find c.

16. Find the amount $\$a$ of a note of $\$p$ which drew simple interest at $r\%$ for n yr. m mo. d da., assuming 12 mo. of 30 da. each to a month. Then find a when $p = 400$, $r = 6$, $n = 3$, $m = 9$, $d = 13$.

17. Find B's tax $\$t$ on $\$p$ worth of property when the total assessed valuation is $\$A$ and the tax to be raised is T. Find t when $p = 4000$, $A = 1{,}946{,}500$, and $T = 33090.50$.

18. Find the insurance premium $\$p$ to be paid on a house valued at $\$h$ when insured for $v\%$ of its valuation at an $r\%$ rate. A carriage factory and stock worth \$40,000 were insured for 90% of their value at $3\frac{1}{2}\%$. Find p.

19. What is the bank discount $\$d$ on a non-interest-bearing note for $\$a$ if discounted at $r\%$, n days before it is due? Find discount $\$d$ on a note for \$300 discounted at 7% 45 days before it was due.

20. A railroad stock is quoted at $n\%$ above par and its annual dividends are $r\%$; what interest rate R is realized by an investor? Suppose $n = 28$, $r = 8$, to find R.

21. What is the duty on n yd. of cloth worth $\$c$ a yard, taxed at $r\%$ ad valorem, plus a specific duty of s¢ a yard? What would be the duty on 75 yd. of cloth worth \$2 a yard, at 55% ad valorem, and 11¢ a yard?

22. The interest on $\$x$ for t yr. and m mo. at a certain rate per cent was $\$i$. What was the rate? Find rate when $t = 2$, $m = 3$, $i = 69.75$, and $x = 620$.

23. The population of a town in 1900 was p and in 1910 it was q. What was the gain per cent r? Make the calculation for the following cities:

	p	q
Cleveland, O.,	381,768	560,663
Atlanta, Ga.,	89,872	154,839
Seattle, Wash.,	80,671	237,194
Los Angeles,	102,479	319,198
New York,	3,437,202	4,766,883

24. A draft on New York was bought at $d\%$ discount for $\$k$. What was its face $\$f$? Find f when $k=480$ and $d=\frac{1}{2}$.

25. When a stock is quoted at $h\%$ below par and pays $d\%$ annual dividends, how much money $\$x$ must be invested to yield the purchaser an income of $\$i$ annually?

Given $h=8,\ d=5,\ i=4000$.

160. Solution of Literal Equations containing Two Unknown Quantities.

1. Given (1) $ax+by=c$; (2) $a'x+b'y=c'$.

SOLUTION

$$\begin{array}{llll}
(2_1) & aa'x + ab'y = ac' & (1_2)\ ab'x + bb'y = b'c & (\text{Ax. ?}) \\
(1_1) & \underline{aa'x + a'by = a'c} & (2_2)\ \underline{a'bx + bb'y = bc'} & (\text{Ax. ?}) \\
 & ab'y - a'by = ac' - a'c & \quad\ \ ab'x - a'bx = b'c - bc' & (\text{Ax. ?})
\end{array}$$

$$y = \frac{ac' - a'c}{ab' - a'b}.\quad \textit{Ans.} \qquad x = \frac{b'c - bc'}{ab' - a'b}.\quad \textit{Ans.}$$

VERIFICATION (1) $\quad a \cdot \dfrac{b'c - bc'}{ab' - a'b} + b \cdot \dfrac{ac' - a'c}{ab' - a'b} = c,$

or, $\quad ab'c - abc' + abc' - a'bc \equiv ab'c - a'bc.$ (Mult. Ax.)

2. (1) $ax - by = 0$; (2) $x + y = c$.

3. (1) $x + ay = b$; (2) $ax - by = c$.

4. (1) $3ax - 2by = c$; (2) $a^2x + b^2y = 5bc$.

5. (1) $p = a + (m-1)d$; (2) $q = a + (n-1)d$; a and d being regarded as the unknowns.

6. (1) $\dfrac{x}{b+c} + \dfrac{y}{a+c} = 2$; (2) $ax - by = ac - bc$.

7. (1) $\dfrac{u}{a+b} + \dfrac{v}{a-b} = 2a$; (2) $u - v = 4ab$.

161. Literal Equations containing more than Two Unknowns.

1. $\begin{cases} (1)\ ax + by = c, \\ (2)\ cx + az = b, \\ (3)\ bz + cy = a. \end{cases}$ (§ 149, 7.)

2. $\begin{cases} ax + by + cz = a, \\ ax - by - cz = b, \\ ax + cy + bz = c. \end{cases}$

162. Problems involving the Solution of Literal Equations.

1. A merchant has two kinds of sugar that he sells at a¢ and b¢ respectively. He desires to make a mixture of c lb. that will sell for d¢ a pound. How many pounds of each kind shall he take?

2. A man distributed a¢ among n persons, some receiving b¢ each, and others c¢ each. How many received b¢ each, and how many c¢?

3. A was m times as old as B a years ago, and will be n times as old as B in b years. Find the age of each at present.

4. A and B can do a piece of work in a days, or if A works m days alone, B can finish the work by working n days. In how many days can each do the work? (See § 146, 9.)

5. A man invested p dollars, part at r %, and the remainder at s %. His annual income from both investments was i dollars. What was the amount of each investment?

6. Two trains are scheduled to leave A and B, d miles apart, at the same time and to meet in h hours. If the train that leaves B is b hours late, and runs at its usual rate, it will meet the other train in k hours. What is the rate of each train?

7. The average age of A, B, and C is m years. The average age of A and B is n years, and of B and C is p years. What are their ages?

CHAPTER VI

INVOLUTION AND EVOLUTION

I. INVOLUTION

163. Involution is the operation of raising quantities to powers.

164. Law for Raising Quantities to Powers. Let it be required to show that $(x^m)^n = x^{mn}$, m and n being integers.

$$(x^m)^n = (xxx \cdots \text{ to } m \text{ factors}) \times (xxx \cdots \text{ to } m \text{ factors}) \times \cdots \text{ to } n \text{ times.}$$
$$= xxxx \cdots \text{ to } mn \text{ times.}$$
$$= x^{mn}.$$

Hence, to raise a quantity with any integral exponent to any integral power, multiply the exponent of the quantity by the index of the power for the exponent of the quantity in the result.

165. Raising Monomials to Powers.

Type form: $(x^a y^b)^n = x^{an} y^{bn}$.

1. Square: $3m^2$; $-5a$; $-6a^m$; $\frac{2}{3}x^3$; $-3a^3b^n$.

2. Cube: $-6xy$; $3abc$; $-2m^2y$; $\frac{2}{5}a^2bc$; $-3a^mb^n$.

3. Simplify: $(3ab)^4$; $(2 \times 3^2 \times 5)^3$; $(3 \times 2a)^4$; $(-\frac{2}{5}bc^n)^3$; $(2^2 \times 5)^4$; $(a^m)^{2p}$; $(a^m b^p c^q)^3$; $(2^2 \times 3^3 \times 5)^n$; $(4^2 \times 5^n \times 7)^a$.

4. Simplify: $3(2x)^2$ (see § 21); $5(2 \times 3a)^4(3b)^2$; $\frac{1}{8}(\frac{1}{2}a)^2(4b)^3$; $6(-3b)^2(-2c)^3$; $2(3 \times 4)^2(2 \times 5)^5$; $2(2a)^n(2b)^m$.

166. Raising Binomials to Powers. Newton's Theorem. By a theorem, or by actual multiplication, show that:

$$(a - b)^2 = a^2 - 2ab + b^2.$$
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$
$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$
$$(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

Examining the right members of these equations, we see;

(1) *That the signs are alternately* $+$ *and* $-$.

(2) *That the exponent of* a *in the first term in each case is the same as the exponent of* $(a - b)$ *in the left member, and that the exponents of* a *decrease by 1 from term to term; also that* b *appears first in each case in the second term, and its exponents increase by 1 from term to term.*

(3) *That the first coefficient in the right member in each case is 1 (understood), and the second coefficient is the same as the exponent of* $(a - b)$ *in the left member; also that each succeeding coefficient can be obtained by multiplying the coefficient of the preceding term by the exponent of the leading letter* a, *and dividing the product by the exponent of the other letter increased by 1.*

1. Raise $(a - b)$ to the 8th power.

SOLUTION. $(a - b)^8 = a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 + 70a^4b^4$
$- 56a^3b^5 + 28a^2b^6 - 8ab^7 + b^8.$

2. $(a - b)^{10}$. **3.** $(m + n)^6$. **4.** $(H - K)^9$.

5. Raise $2a^2 - 3b$ to the fifth power.

SOLUTION. $(x - y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$.

Then, $(2a^2 - 3b)^5 = (2a^2)^5 - 5(2a^2)^4(3b) + 10(2a^2)^3(3b)^2$
$- 10(2a^2)^2(3b)^3 + 5(2a^2)(3b)^4 - (3b)^5$

(See § 21, 1.) $= 32a^{10} - 240a^8b + 720a^6b^2 - 1080a^4b^3 + 810a^2b^4$
$- 243b^5.$

6. $(3m^2 - 5n)^3$. **7.** $(2m^2n^3 - 1)^6$. **8.** $(4m^n - \frac{1}{3}p^3)^4$.

9. $(10 - 1)^4$. **10.** $(14\frac{1}{4})^3 = (14 + \frac{1}{4})^3$. **11.** $(12\frac{1}{6})^3$

In cases of raising a sum or difference of two quantities of which one is small as compared with the other, Newton's theorem furnishes a quickly and easily obtained *approximate* result, since after the first or second terms, the numerical values of the succeeding terms diminish rapidly and some or all of the latter terms can be neglected. Thus,

$$(122\tfrac{1}{3})^3 = 122^3 + \cancel{3} \times 122^2 \times \frac{1}{\cancel{3}} + 3 \times 122 \times \frac{1}{9} + \frac{1}{27}$$
$$= 122^2(122 + 1) + 41 \text{ nearly, or } 1{,}830{,}773.$$

12. $(100-\frac{1}{2})^3$. 13. $(18\frac{1}{6})^3$. 14. $(2\frac{1}{16})^4$.

15. $(5\frac{19}{20})^5=(6-\frac{1}{20})^5$. 16. $(1\frac{11}{12})^6$. 17. 999^4.

167. Raising Polynomials to Powers. This is accomplished by changing the given polynomial into a binomial by the use of parenthesis, and applying Newton's theorem.

1. Cube $2a+3b-c$.

SOLUTION. $(2a+3b-c)^3=[(2a+3b)-c]^3$

$$=(2a+3b)^3-3(2a+3b)^2c+3(2a+3b)c^2-c^3.$$

The solution is continued by expanding each term of this result, then multiplying and combining terms when this can be done.

2. $(m^2-3mn+4n^2)^3$. 3. $(a^m+a^n-2c^{2p})^3$.

4. *Indicate* the expansion of $(2x-3y+4z-5n)^4$ without expanding the several terms.

II. EVOLUTION

168. Evolution is the process of extracting roots, that is, of finding equal factors of quantities.

169. Law for Extracting Roots.

Let it be required to show that $\sqrt[q]{x^{pq}}=x^p$.

We have, $x^{pq}=(x^p)^q$. (By § 164.)

Then, $\sqrt[q]{(x^p)^q}=x^p$. (Since the qth root of the qth power of a quantity equals that quantity, by the definitions of power and root, §§ 12, 13.)

Hence, to extract a root of a quantity, divide the exponent of the quantity by the index (§ 13) *of the root for the exponent of the quantity in the answer.*

170. Extraction of the Roots of Monomials.

Type form: $\sqrt[n]{a^{mn}b^{np}}=a^mb^p$.

Even roots of positive quantities can have for sign either + or − and are marked ± (see § 44, 17). Even roots of negative quantities cannot be real quantities. Thus, $\sqrt{-4x^2}$ is not $2x$, since $(2x)^2=+4x^2$; neither is it $-2x$, since $(-2x)^2$ also equals $+4x^2$, and not $-4x^2$. Even roots of negative quantities have been called **imaginaries.**

1. $\sqrt{36\,a^2}$.
2. $\sqrt[3]{-64\,a^3b^6}$.
3. $\sqrt[5]{32\,a^{10}}$.
4. $\sqrt[m]{a^{2m}b^{5m}c^{3m}}$.
5. $\sqrt{8 \times 3^3 \times 6}$.
6. $\sqrt[3]{-2^6 \times 5^3 \times 7^9}$.
7. $\sqrt[4]{\dfrac{16\,a^4}{81\,b^8}}$.
8. $\sqrt[10]{\dfrac{x^{10}y^{20}}{z^{30}}}$.
9. $\sqrt{\dfrac{5 \times 3^2 \times 2^8}{2^2 \times 5^{13} \times 3^5}}$.

171. Square Root of Polynomials.

Let it be required to derive a process for the extraction of the square root of a polynomial from a study of the formula

$$a^2 + 2\,ab + b^2 = (a + b)^2.$$

SOLUTION.

$$\begin{array}{r|l}
 & a^2 + 2ab + b^2 \;\; \underline{|\; a + b} \\
 & a^2 \\
\hline
2a + b & 2ab + b^2 \\
 & \underline{2ab + b^2}
\end{array}$$

How is a, the first term of the root, found? How is $2\,a$, the trial divisor, found from the first term of the root? How is b, the second term of the root, found? In what two places is b written when found? How is the operation concluded?

1. Extract square root of $4\,x^4 - 12\,x^3 + 5\,x^2 + 6\,x + 1$.

SOLUTION.

$$\begin{array}{r|l}
 & 4x^4 - 12x^3 + 5x^2 + 6x + 1 \;\; \underline{|\; 2x^2 - 3x - 1 \;\; \textit{Ans.}} \\
 & 4x^4 \\
\hline
4x^2 - 3x & -12x^3 + 5x^2 \\
 & \underline{-12x^3 + 9x^2} \\
(4x^2 - 6x) - 1 & -4x^2 + 6x + 1 \\
 & \underline{-4x^2 + 6x + 1}
\end{array}$$

REMARK. After two terms of the root are found, to find the next trial divisor, the two terms are regarded as one quantity or as a monomial and are doubled, giving $4\,x^2 - 6\,x$.

2. $81\,x^4 - 432\,x^3 + 864\,x^2 - 768\,x + 256$.

3. $1 - 2\,z + 2\,z^2 - z^3 + \dfrac{z^4}{4}$.

4. Extract fourth root, *i.e.* square root twice, of $a^4 + 8\,a^3b + 24\,a^2b^2 + 32\,ab^3 + 16\,b^4$.

5. Make a rule for the extraction of the square root of algebraical quantities, mentioning first the matter of arrangement of terms before beginning.

6. $x^6+25x^2+10x^4-4x^5-20x^3+16-24x.$

7. $\sqrt{x(x+1)(x+2)(x+3)+1}=?$

Solve the two following mentally by getting the first and last terms, and then the middle one. Test the answer carefully, paying special attention to the sign of the last term.

8. $\sqrt{x^4-2x^3+3x^2-2x+1}=?$

9. $\sqrt{9x^4-12x^3+16x^2-8x+4}=?$

172. Extraction of the Square Root of Arithmetical Numbers.

1. Extract the square root of 1156.

Solution. Since 1156 lies between 900 and 1600, its square root lies between 30 and 40, *i.e.* between 3 tens and 4 tens.

In general counting from the unit's order, each pair of figures, or "period," in the number gives one figure in the root, except that the left-hand period may have only one figure. Thus, the square root of 1'49 is 12+. The periods are usually marked by little lines above and between the figures.

For guidance in the solution the following formula is used:

$$(t+u)^2=t^2+2tu+u^2=t^2+(2t+u)u.$$

Notice in the last expression that $2t$ is the *trial* divisor, and $2t+u$ is the *complete* divisor. The process at the right below is just like that at the left. That at the left was explained in the last article.

$$\begin{array}{r|l}
\multicolumn{2}{l}{t^2+2tu+u^2\ \underline{|\ t+u}}\\
\multicolumn{2}{l}{\underline{t^2\qquad\qquad}}\\
2t+u & 2tu+u^2\\
 & \underline{2tu+u^2}
\end{array}
\qquad\qquad
\begin{array}{r|l|l}
\multicolumn{3}{r}{t+u}\\
\multicolumn{2}{r}{11'56} & \underline{30+4}\\
\multicolumn{2}{r}{t^2=\quad \underline{9\,00}} & \\
\text{Trial divisor }=2t=60 & 2\,56 & \\
u=4 & & \\
\hline
\text{Complete divisor }=2t+u=64 & \underline{2\,56} &
\end{array}$$

2. Extract the square root of 43,347.24 and prove answer.

Solution.

$$\begin{array}{r|l}
\multicolumn{2}{l}{4'33'47'.24\ \underline{|\ 208.2}}\\
\multicolumn{2}{l}{\underline{4\qquad\quad}}\\
408 & 33\,47\\
 & \underline{32\,64}\\
4162 & 83\,24\\
 & \underline{83\,24}
\end{array}
\qquad (208.2)^2=43347.24 \text{ by actual multiplication.}$$

Explanation. In the solution to Ex. 1, the ciphers were retained to make the process clear. In this solution no cipher is written that is not essential to the solution.

When the first figure of the root is doubled for trial divisor it is ***understood*** to be 40, though no zero is written after 4 at first. Now 40 is not contained in the period 33 brought down, so 0 is written in the root and after 4, and a new period, 47, is brought down. Next, 40 is understood to be 40 tens or 400. This 400 is contained in 3347 eight times. When the decimal period is brought down, the decimal point is inserted in the root.

3. 14,356,521. **4.** 33,790,969. **5.** 16,803.9369.

Solve the following, getting three decimal places in the root, and prove answers by squaring the root and adding in remainder thus getting given number.

6. 2.5. **7.** $\frac{1}{70}$. **8.** .008. **9.** .1.

a. Every sequence of figures has two square roots due to the position of the decimal point. Thus $\sqrt{25} = 5$; $\sqrt{2.5} = 1.6^-$. This shows importance of pointing off number into periods correctly, commencing at decimal point. Notice $\sqrt{.5}$ must be written $\sqrt{.50}$ to get results correct. It is important to check the square root of decimals by multiplication, especially when first beginning the subject.

10. Make a rule for extracting square root of arithmetical numbers, explaining carefully exceptional cases.

11. 2. **12.** .056. **13.** $\frac{2}{7}$. **14.** .00003.

Solve following mentally, getting one decimal place in root:

15. 2. **16.** 3. **17.** .5. **18.** 209.

***173. Extraction of Cube Root of Algebraic Quantities.** To derive a process for the extraction of the cube root of polynomials from the formula:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + (3a^2 + 3ab + b^2)b.$$

SOLUTION

$$\begin{array}{r|l} & a^3 + 3a^2b + 3ab^2 + b^3 \quad \underline{a + b} \\ & a^3 \\ \hline 3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\ & \underline{3a^2b + 3ab^2 + b^3} \end{array}$$

How is first term of root found? How is trial divisor $3a^2$ found from first term of root? How is second term of root found? What two terms are added (annexed) to the trial divisor to make complete divisor? Explain precisely how they are obtained. How is process completed? Is it important that the terms be arranged before beginning?

---

*This subject is often omitted. It is not required for college entrance.

1. $8x\backslash + 66x^4 - 63x^3 + 33x^2 - 9x + 1 - 36x^5.$

SOLUTION

$$\begin{array}{rl}
 & 8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1 \,\underline{|\, 2x^2 - 3x + 1} \\
 & \underline{8x^6 \qquad\qquad\qquad\qquad\qquad\qquad} \\
12x^4 - 18x^3 + 9x^2 \,| & -36x^5 + 66x^4 - 63x^3 \\
| & \underline{-36x^5 + 54x^4 - 27x^3} \\
12x^4 - 36x^3 + 27x^2 \qquad\qquad | & 12x^4 - 36x^3 + 33x^2 - 9x + 1 \\
6x^2 - 9x \qquad | & \\
+1 \,| & \\
\overline{12x^4 - 36x^3 + 33x^2 - 9x + 1} \,| & \underline{12x^4 - 36x^3 + 33x^2 - 9x + 1}
\end{array}$$

EXPLANATION. 1st trial divisor $= 3(2x^2)^2$.

2d trial divisor $= 3(2x^2 - 3x)^2$.

$3(2x^2 - 3x) \times 1 = 6x^2 - 9x.$

$1^2 = 1.$

2. $x^3 - 24x^2 + 192x - 512.$

3. $x^6 - 3x^5y + 6x^4y^2 - 7x^3y^3 + 6x^2y^4 - 3xy^5 + y^6.$

4. $8x^6 + 48cx^5 + 60c^2x^2 - 80c^3x^3 - 90c^4x^2 + 108c^5x - 27c^6.$

5. Make a rule for finding the cube root of algebraical quantities.

6. $39x^4 - 99x^3 - 9x^5 + x^6 + 64 - 144x + 156x^2.$

7. Find the first three terms of the cube root of $1 - x$. Prove by cubing root (§ 167) and adding remainder.

***174. Extraction of the Cube Root of Arithmetically Expressed Numbers.**

1. Extract cube root of 405,224.

SOLUTION. Since 405,224 lies between 343,000, which is the cube of 70, and 512,000, which is the cube of 80, the cube root of 405,224 lies between 7 tens and 8 tens. In general, counting from unit's order, each set of three figures or "period" in the number gives one figure in the root, except that the left-hand period may have three or two figures or only one figure. Thus, the cube root of 1 728 is 12.

---

* This subject is often omitted. It is not required for college entrance.

Solution with Explanation

$$\begin{array}{rr|l}
 & & t+u \\
 & 405'224 & \underline{70+4} \\
t^3 = & \underline{343\,000} & \\
3\,t^2 = & 14700 & 62\,224 \\
3\,tu = & 840 & \\
u^2 = & \underline{16} & \\
 & 15556 & \underline{62\,224}
\end{array}$$

Regular form of Solution

$$\begin{array}{rr|l}
 & 405'224 & \underline{74} \\
 & \underline{343} & \\
3\times 70^2 = & 14700 & 62\,224 \\
3\times 70\times 4 = & 840 & \\
4^2 = & \underline{16} & \\
 & 15556 & \underline{62\,224}
\end{array}$$

2. Extract the cube root of 232435.510 to one decimal place and prove answer.

Solution.

$$\begin{array}{r|l}
232'435.'510 & \underline{61.4} \\
\underline{216} & \\
10800 & 16\,435 \\
180 & \\
\underline{1} & \\
10981 & \underline{10\,981} \\
1116300 & 5454.510 \\
7320 & \\
\underline{16} & \\
1123636 & \underline{4494.544} \\
 & 959.966
\end{array}$$

Explanation

$3\,t^2 = 3 \times 60^2 = 10800,$
$3\,tu = 3 \times 60 \times 1 = 180,$
$u^2 = 1^2 = 1,$
$3\,t^2 = 3 \times 610^2 = 1116300,$
$3\,tu = 3 \times 610 \times 4 = 7320,$
$u^2 = 4^2 = 16.$

Check. $(61.4)^3 + 959.966 = 232435.510.$

• The answer to the nearest tenth is 61.5, since the next figure of the root after 4 would be more than 5.

3. 12,167. 4. 12,812.904. 5. 167.284151.

In the following get 2 places in the root and verify:

6. 3. 7. 0.2. 8. $\frac{1}{254}$.

9. Write out a rule for cube root, including peculiarities.

In following get answer to nearest tenth mentally:

10. 10. 11. 37. 12. 150.

175. Symmetry in Algebra. A quantity is **symmetrical** if the letters in it can change places without changing its value.

Thus, $a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$ is symmetrical, as is the sum of a and b raised to any integral power, since a and b can interchange without altering the value of the expression.

Similarly, $x^2 + y^2$, $x^2 \pm xy + y^2$, $\frac{a^2+b^2}{2\,ab}$, etc., are all symmetrical.

III. ABRIDGED MULTIPLICATIONS AND DIVISIONS

***176. Abridged Multiplication.** Every one who has used the regular process of multiplication of decimals has felt at times that a portion of the work was wasted.

In the Arabic language, as in the Hebrew, the writing extends from right to left. This accounts for calculations in the Arabic notation proceeding from right to left instead of from left to right as in our reading. But there is no reason why we cannot begin multiplying by the highest order instead of by the lowest.†

1. Multiply 172.181 by 163.56, getting the product correct to one decimal place.

Solution

```
   172‸1ˎ8ˎ1
   163.56
 ----------
 17218.1
 10330.86
   516.54
    86.09
    10.33
 ----------
 28161.9
```

Explanation. The multiplier is set under the multiplicand with one decimal point under the other. We multiply first by 1, which is here 100ˊ. This moves the decimal point in the product two places to the right of where it now is in multiplicand, bringing the right-hand figure of the product in tenths' place. Next we multiply by 6 (60), starting this product one place to the right of the product just set down.

Now, to get tenths in the answer correct, hundredths should be kept. But, multiplying by 3, one gets thousandths, which figure need not be written down. We therefore insert a little mark before the last figure in the multiplicand, and say, 3 × ˎ1 = .3, giving nothing to carry. Then, 3 × 8 = 24; the 4 is set down under 6, and from there on it is ordinary multiplication. When we multiply by the next figure, 5, another figure is cut off in the multiplicand, and we say, 5 × ˎ8 = 4.0, giving 4 to carry; 5 × 1 = 5; 5 and 4 to carry are 9. This 9 is set down under 4 in hundredths' order; and so on.

In adding we do not write the hundredths in the answer, but merely find the number to the nearest ten to carry from the last column.

2. Rule for abridged multiplication.

(1) *Set the multiplier under the multiplicand as if for addition.*

---

* This subject is treated here because it is not found in many arithmetics. Pupils should accustom themselves to performing multiplication of decimals as here explained, because time is saved and the process is more rational than the usual one.

† In the "Encyclopædia Britannica" (11th Edition) under the title *Arithmetic*, will be found a discussion of this question of the order in multiplication.

(2) *Multiply first by the left-hand figure of the multiplier. If the first multiplier figure is units, set the right-hand figure of the partial product directly under the right-hand figure of the multiplicand. If the first figure is not units, move the product to the left or right as many places as the figure is to the left or right of units, and proceed in the same way with the other multiplier figures.*

(3) *One more decimal place in the product should be kept than is asked for in the answer. As soon as a multiplier figure gives an order lower than the lowest to be kept, the right-hand figure of the multiplicand is cut off by a little mark, and when this figure is multiplied by the multiplier figure, the product is not set down, but the tens, to the nearest unit, are carried, and added with the product of the next figure of the multiplicand by this multiplier figure.*

In the next partial multiplication, another figure is cut off at the right of the multiplicand, and the multiplication proceeds in the same way; and so on.

(4) *In adding, the sum in the right-hand column is not set down, but the nearest number of tens is carried.*

3. Multiply 6.7421 by 4.7825, getting tenths correct. Check by ordinary multiplication.

4. Multiply 6.9321 by 4.7825, getting hundredths correct. Check.

5. Multiply 65.324 by 63.412, getting tenths correct. Check.

6. Multiply 264.72 by 34.621, getting tenths correct. Check.

7. If the pupil will now compare the usual solutions with the abridged solutions, he will see that he can draw a vertical line through the usual solution cutting off one part and having left the abridged solution, but with its right column somewhat changed by carrying.

8. Find the compound interest, correct to the nearest cent, on $729.38 for four years at 6% if the interest on $1 as given in a table is $.262477.

9. Find the compound interest amount on $867.93 for 7 yr. 6 mo. if the rate is 5% and the interest is compounded semi-

annually, if the compound interest table gives $1.448298 as the amount on $1 for the given time at the given rate.

10. Find the cost correct to cents of 41,654 ft. of lumber at $18.75 per M.

11. Find the cost of 6347 lb. of coal at $7.25 a ton.

177. Abridged Division. 1. A stock that pays 7.5% annual dividends, that is, $7.50 per share of $100, costs, including brokerage, $164.375. Find the rate of interest realized correct to four decimal places.

Solution

$$\begin{array}{r}
.0456\\
\text{16ˎ4ˎ3ˎ75})\overline{7500.00}\\
\underline{6575}\\
925\\
\underline{822}\\
103\\
\underline{98}
\end{array}$$

Explanation. The decimal point is moved three places to the right in both divisor and dividend, which is equivalent to multiplying each by 1000, thus leaving the quotient unaltered. This makes the divisor a whole number, places the decimal point in the quotient immediately over that in the dividend, and the first significant figure of the quotient over the last figure of the dividend used to get it when all figures of the divisor are used.

The order of the first significant figure in the quotient is thus hundredths. This leaves two figures of the quotient still to be found. If we wish 2 figures only to appear in the divisor at the end of the division, and one figure is cut off from the divisor each time as we get the two remaining figures of the quotient, then four figures will be all we shall need in the divisor when we begin the division. Consequently 75 is cut off, thus, ˎ75, before beginning to divide.

The operation of dividing proceeds as follows: $4 \times .7 = 2.8$, giving 3 to the nearest unit to carry; then $4 \times 3 + 3 = 15$, the 5 being set down and 1 carried; then the operation continues as usual. In this way we get 6575.

To get the next figure of the quotient, the 3 in the divisor is now cut off by a mark. Then we say $5 \times .3 = 1.5$, giving 2 to carry; $5 \times 4 + 2 = 22$, 2 being set down and 2 carried; in this way we get 822. In the last operation, 4 is cut off, and we begin by saying $6 \times .4 = 2.4$, which gives 2 to carry; then $6 \times 6 + 2 = 38$, or 8 down and 3 to carry; $6 \times 1 + 3 = 9$.

2. **Rule for Abridged Division.** (1) *Make the divisor a whole number without zeros at its right, by moving the decimal point. Move the decimal point in the dividend the same number of places and in the same direction as in the divisor.*

(2) *Place the decimal point in the quotient immediately over that in the dividend, and the first significant figure in the quotient over the last figure of the dividend used to get it if all the figures of the divisor were used.*

(3) *To the number of figures in the quotient yet to be found add 2. This gives the number of figures needed in the divisor. Cut off with mark any excess over this at the right of the divisor. If there is no such excess at the beginning of the operation, test for such excess as you proceed, using the rule just given.*

(4) *Each time multiply the left-hand figure of those cut off by the figure of the quotient and carry the tens of the product to the nearest unit, but do not set down the units. In each successive division thereafter cut off a new figure at the right of the divisor.*

3. Divide 163.27 by 48.613, getting three decimal places in the quotient.

4. Divide 267.46 by 23,740, getting three decimal places in quotient.

5. Find the present worth of $258.19 due in 9 mo. and 14 da. at 6 %, that is, divide 258.19 by 1.04733^+, getting answer correct to cents.

6. A man bought a stock of goods for $12,674.29 and sold it for $14,695.18. What per cent, to the nearest tenth, did he gain?

7. How many bushels, to the nearest hundredth, are there in 66,374 cu. in., if 1 bu. contains 2150.42 cu. in.?

8. What principal, to the nearest cent, will amount to $756.25 in 2 yr. 8 mo. 17 da., at 6 %?

9. What is the diameter of a circle, to the nearest tenth, whose circumference is 65.8 in., if π is taken equal to 3.14? If 65.8324 is circumference, and π is taken equal to 3.1416?

10. What is the width of a rectangle whose length is 16.8 rd., and whose area is 818.19 sq. yd., getting answer to nearest tenth of a yard?

IV. PRACTICAL APPLICATION OF INVOLUTION, EVOLUTION, AND ABRIDGED MULTIPLICATION AND DIVISION

178. Solution of Problems by the Use of Involution, Evolution, Multiplication, and Division. Formulas.

1. Calculate in the quickest way $2^2 \times 3^2 + 2 \times 3^2 + 2^3 \times 3^3 + 2 \times 3^4$ by taking out the common factor.

2. Calculate $18^2 + 15^2 + 6^3 + 9^3 + 36^2$ by changing each quantity into the product of its prime factors with exponents and then taking out the common factor. Thus, $18^2 = (2 \times 3^2)^2 = 2^2 \times 3^4$; etc.

3. Write as the product of their prime factors raised to powers, each of the following: 54^3; 125^2; 24^4; 108^3; 36^5.

4. $(a + b + c)^2 + (a - b - c)^2 = ?$ See § 68.

5. $(x + y - z)^2 - (x - y + z)^2 = ?$ 6. $(a - b)^2(b - a)^3 = ?$

7. Find the area of a square whose side is 6.05 ft., correct to tenths. See § 63.

8. Find the volume of a cube whose edge is $8\frac{1}{8}$ in., correct to two decimal places. § 166, 11.

9. Find the area, correct to tenths, of a circle whose radius is $11\frac{1}{2}$ in. from the formula $a = \pi r^2$, using $\pi = 3.1416$. (See § 88 for square of number $+ \frac{1}{2}$.)

10. Find the volume, correct to hundredths, of a sphere whose diameter is 4.02 from the formula $v = \frac{1}{6}\pi d^3$.

11. Find the surface, correct to hundredths, of a sphere whose radius is $3\frac{1}{2}$ m., from the formula $s = 4\pi r^2$.

12. Find the volume correct to hundredths of a cylinder whose altitude is 3.875 ft. and radius of base 2.268 ft., from the formula $v = \pi r^2 h$, in which r is the number of feet in the radius of base and h the number of feet in the altitude.

13. Find the volume correct to hundredths of a cone whose altitude a is 20.5 cm. and radius of its base 144 cm., from the formula $v = \frac{1}{3}\pi a r^2$.

14. Find the area correct to hundredths of the triangle whose three sides a, b, c are respectively 8, 9, 13 in. from the formula

$$A=\sqrt{s(s-a)(s-b)(s-c)}, \quad \text{(See § 89.)}$$

in which

$$s=\tfrac{1}{2}(a+b+c).$$

Solution. $s=\frac{1}{2}(8+9+13)=15$; $s-a=7$; $s-b=6$; $s-c=2$.

Then $A=\sqrt{15\times7\times6\times2}=\sqrt{1260}=35.5^{-}$. *Ans.*

Check. $35.5^2=1260.25$.

15. Find the areas correct to hundredths of the following triangles in same way: $a=10$, $b=12$, $c=14$; $a=1.6$, $b=1.9$, $c=2.1$; $a=346$, $b=579$, $c=722$; $a=.06$, $b=.053$, $c=.019$.

16. Find the hypotenuse correct to hundredths of a right triangle whose two legs are 6.21 ft. and 9.42 ft. from the formula $h=\sqrt{a^2+b^2}$.

17. Find one leg of a right triangle whose hypotenuse is $7\frac{1}{2}$ in. and other leg is $4\frac{1}{4}$ in. from the formula $a=\sqrt{h^2-b^2}$.

18. Find the time of falling correct to tenths of a second from the formula $t=\sqrt{\frac{2s}{g}}$, knowing the distance to be 168.2 ft. and $g=32.2$ ft.

19. Find the surface correct to two decimal places of a cube whose volume is 168.2 cu. ft.

20. Find the surface of a sphere whose volume v is 19 cu. ft. from the formula $s=\sqrt[3]{36\pi v^2}$.

21. The volume v of a spherical segment or part of sphere included between two circles in parallel planes whose radii are r and r' and whose altitude h is the perpendicular distance between the parallel planes is given by the formula

$$v=\frac{h}{2}(\pi r^2+\pi r'^2)+\frac{\pi h^3}{6}.$$

Find v when $r=6.1$, $r'=8.2$, and $h=1.08$. (See § 166, 11.)

22. Find the length of a perpendicular from the right angle to the hypotenuse of a right triangle if it divides the hypotenuse which is 10 ft. long into segments one of which is .5 ft. longer than the other.

Suggestion. Solve first for x and y, the segments of the hypotenuse. See § 51.

23. By measurement the distances on a line drawn from a point outside a circle to it are 12.7 ft. to the nearest point of the circumference and 19.6 ft. to the farther point on the circumference. What is the length on the tangent to the circle from the given point measured from the point of tangency to the given point?

24. If AB is the chord of a circle whose radius is R, and AD is the chord of half the arc AB, then, by geometry,

$$AD=\sqrt{2R-R\sqrt{4R^2-\overline{AB}^2}}.$$

If the radius of a circle is 8 ft. and the chord AB is 2 ft., find AD.

25. If AB of preceding Exercise is one side of a regular hexagon inscribed in a circle whose radius is 1 ft., then AD is one side of a regular inscribed 12-agon. Calculate the *perimeter* of the regular 12-agon after first calculating one of its sides by means of the formula in Ex. 24. (See § 51, 21.)

26. Using the value of AD found in Ex. 25 calculate one side of a regular 24-agon inscribed in a circle whose radius is 1 ft., and then find its perimeter.

27. Explain, now, how the formula of Ex. 24 can be used to find an approximate value of π.

CHAPTER VII

FRACTIONAL EXPONENT QUANTITIES AND RADICALS.*

I. FRACTIONAL EXPONENT EXPRESSIONS

179. Formulas Giving the Laws for Integral Exponents.

1. Addition and subtraction formulas:

$ax^m + bx^m = (a+b)x^m$; $ax^m - bx^m = (a-b)x^m$. (§ 35.)

2. Multiplication formula: $x^m \times x^n = x^{m+n}$. (§ 42.)

3. Division formula: $x^m \div x^n = x^{m-n}$; $\frac{x^m}{x^n} = x^{m-n}$. (§ 48.)

4. Power formulas: $(x^m)^n = x^{mn}$; $(xyz)^m = x^m y^m z^m$. (§ 164.)

5. Root formulas: $\sqrt[n]{x^m} = x^{\frac{m}{n}}$; $\sqrt[n]{x^m y^p} = x^{\frac{m}{n}} y^{\frac{p}{n}}$. (§ 169.)

180. Meaning of Fractional Exponents. Integral exponents were defined in § 11. We are now to determine a meaning for fractional exponents. Such definition will naturally have to agree with the definition and formulas for integral exponents, since integral exponents are merely special cases of fractional exponents having the denominator 1.

If we let the laws of § 164 and § 41 hold true for fractional exponents, we have:

$$(x^{\frac{1}{2}})^2 = x^{\frac{2}{2}} = x;\ x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x.$$

Thus, $x^{\frac{1}{2}}$ *is the square root of* x, since it is *one* of two equal factors whose product is x. (See § 13.)

Again, $(x^{\frac{1}{3}})^3 = x^{\frac{3}{3}} = x$; $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x^1 = x$.

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* Throughout this chapter many problems can be solved mentally and should be so solved.

Thus, $x^{\frac{1}{3}}$ *is the cube root of* x, since it is *one* of three equal factors whose product is x.

In general, $(x^{\frac{m}{n}})^n = x^{\frac{mn}{n}} = x^m$;

$$x^{\frac{m}{n}} \times x^{\frac{m}{n}} \times x^{\frac{m}{n}} \times \cdots \text{ to } n \text{ factors} = x^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\text{to } n \text{ terms}} = x^{\frac{mn}{n}} = x^m.$$

Thus, $x^{\frac{m}{n}}$ is the *nth* root of x^m, since it is *one* of n equal factors whose product is x^m.

We are therefore led to attach the following meaning to a fractional exponent:

The numerator of a fractional exponent denotes the power to which the quantity of which it is the exponent is to be raised, and the denominator denotes the root to be taken of this result.

The student will see this more clearly from the following examples; $\sqrt{a^4} = a^{\frac{4}{2}} = a^2$; $\sqrt[3]{x^6} = x^{\frac{6}{3}} = x^2$; $\sqrt[4]{b^{12}} = b^{\frac{12}{4}} = b^3$; then $\sqrt[3]{a^7} = a^{\frac{7}{3}}$, and $a^{\frac{7}{3}}$ means the cube root of the 7th power of a.

181. Exercise dealing with Quantities having Fractional Exponents.

1. Calculate $8^{\frac{2}{3}}$.

Solution. $8^2 = 64$; $\sqrt[3]{64} = 4$. *Ans.* Or, $\sqrt[3]{8} = 2$; $2^2 = 4$. *Ans.*

Evidently it is easier to extract the root *first*.

Another Solution. $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$. *Ans.*

Here the number is first expressed as the power of a prime.

a. To extract the root of a fraction extract the root of each term.

$$\text{Thus, } \sqrt{\frac{4}{9}} = \frac{2}{3}. \quad \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}.$$

2. Calculate: $4^{\frac{1}{2}}$; $16^{\frac{1}{4}}$; $27^{\frac{2}{3}}$; $4^{\frac{3}{2}}$; $9^{\frac{3}{2}}$; $(\frac{1}{8})^{\frac{2}{3}}$; $(-\frac{1}{27})^{\frac{2}{3}}$; $(\frac{8}{27})^{\frac{4}{3}}$; $(a^m)^{\frac{1}{m}}$; $(x^{pq})^{\frac{1}{q}}$.

3. Calculate: $16^{\frac{1}{4}}$; $16^{\frac{1}{2}}$; $49^{\frac{3}{2}}$; $(36)^{\frac{5}{2}}$; $(\frac{1}{4})^{\frac{7}{2}}$.

4. $(9\,m^4 - 30\,m^2n + 25\,n^2)^{\frac{1}{2}} = ?$

5. $(a^3 - 6\,a^2b + 12\,ab^2 - 8\,b^3)^{\frac{1}{3}} = ?$

6. Express with fractional exponents: $\sqrt[5]{x^4}$ (*Ans.* $x^{\frac{4}{5}}$); $\sqrt[3]{a^2}$; $\sqrt[6]{m^{11}}$; $\sqrt[m]{y^n}$; $\sqrt[3]{m^2}$.

7. Express with radical signs: $x^{\frac{4}{3}}$; $a^{\frac{3}{4}}$; $p^{\frac{1}{2}}$; $3\,z^{\frac{1}{3}}$.

8. Calculate $243^{\frac{2}{5}}$; $36^{\frac{3}{2}}$; $9^{\frac{5}{2}}$; $81^{\frac{3}{4}}$.

182. Principles underlying Operations with Quantities containing Fractional Exponents.

We know that $(abc)^2 = a^2b^2c^2$; $(abc)^3 = a^3b^3c^3$; etc. The question arises, is it likewise true that

$$(abc)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}};\ (abc)^{\frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}};\ \text{etc.?}$$

Or, stated generally, is it true that

$$(1) \qquad (abc\cdots)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots?$$

Now we can test the truth of this equation by raising both sides to the nth power by the power axiom (§ 57). We have

$$[(abc\cdots)^{\frac{1}{n}}]^n = abc\cdots. \text{ (By the definitions of root and power.)}$$

and

$$(a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots)^n = (a^{\frac{1}{n}})^n(b^{\frac{1}{n}})^n(c^{\frac{1}{n}})^n\cdots$$
$$= abc\cdots. \text{ (By the definitions of root and power.)}$$

Thus, we see that raising the two members of equation (1) to the same power gives equals. This, however, does not *prove* that $(abc\cdots)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots$, since in algebra unequals raised to the same power may give equals. For example, $(-2)^2 = +4$, and $(+2)^2 = +4$; but this does not prove that -2 equals $+2$, for they are *not* equal.

If, however, we limit the roots considered to arithmetical values, that is, to positive numbers, then it is always true that if two numbers raised to the same power give equals, they are equal.

Hence, with the limitation in meaning stated, we have

$$(abc\cdots)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots.$$

Changing this formula into a principle, we obtain:

1. **Fundamental Principle.** *The arithmetical root of a product is equal to the product of the same arithmetical roots of the several factors, and conversely.*

By means of this principle any **factor** within a sign denoting a root may be removed outside the sign provided the desired root of the factor can be extracted.

For example, $12^{\frac{1}{2}} = (4 \times 3)^{\frac{1}{2}} = 4^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 2(3)^{\frac{1}{2}}$.

By means of this principle also like roots of quantities can be multiplied together. Thus, $6^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 30^{\frac{1}{2}}$.

2. *If both terms of a fractional exponent are multiplied by the same number, the value of the quantity is not changed.*

This principle holds, likewise, only for arithmetical roots.

For, while $9^{\frac{1}{1}} = 9$, $9^{\frac{2}{2}}$ may equal either $+9$ or -9, since the square root (denoted by the denominator 2 of the exponent) may be either positive or negative.

3. *A quantity with a fractional exponent may have its value calculated either by raising to the power first and extracting the root afterwards, or vice versa.* See Ex. 1, § 181.

This principle also holds true only for arithmetical roots.

Thus, $25^{\frac{2}{2}} = (625)^{\frac{1}{2}} = \pm 25$, while $(25^{\frac{1}{2}})^2 = (\pm 5)^2 = +25$ only.

183. Proofs based on principles of § 182 to show that same laws govern the use of fractional exponents as governed integral ones.

1. *Proof that in multiplication the exponents of like factors are added.*

Let $x^{\frac{m}{n}}$ and $x^{\frac{p}{q}}$ be any two quantities with fractional exponents, x being any quantity, and m, n, p, q being integral numbers.

Then,
$$\begin{aligned} x^{\frac{m}{n}} \times x^{\frac{p}{q}} &= x^{\frac{mq}{nq}} \times x^{\frac{np}{nq}} && (\S\ 182,\ 2.) \\ &= (x^{mq})^{\frac{1}{nq}} \times (x^{np})^{\frac{1}{nq}} && (\S\ 182,\ 3.) \\ &= (x^{mq} \times x^{np})^{\frac{1}{nq}} && (\S\ 182,\ 1.) \\ &= (x^{mq+np})^{\frac{1}{nq}} && (\S\ 42.) \\ &= x^{\frac{mq+np}{nq}}. && (\S\ 182,\ 3.) \end{aligned}$$

But $\frac{mq+np}{nq}$ is the **sum** of the exponents $\frac{m}{n}$ and $\frac{p}{q}$.

Hence the rule to add the exponents of the same letter in the factors for the exponent of this letter in the product holds for fractional as well as for integral exponents.

2. Derive the corresponding rule for division. (See § 48.)

3. *Proof that in raising a quantity to a power the exponent, of the quantity is multiplied by the index of the power for the exponent of the quantity in the result.*

Let $a^{\frac{m}{n}}$ be the given quantity and $\frac{p}{q}$ be the exponent, a being any quantity and m, n, p, q being integral numbers.

Then $$(a^{\frac{m}{n}})^{\frac{p}{q}}=([(a^m)^{\frac{1}{n}}]^{\frac{1}{q}})^p$$ (By § 182, 3.)

$$=[(a^m)^{\frac{1}{nq}}]^p$$ (Since the qth root of the nth root means the nqth root by the definition of root, § 13, the nqth root being one of q equal factors of *one* of the n equal factors of a^m.)

$$=[(a^m)^p]^{\frac{1}{nq}}$$ (By § 182, 3.)

$$=(a^{mp})^{\frac{1}{nq}}$$ (Since the pth power of the mth power is the mpth power, § 164.)

$$=a^{\frac{mp}{nq}}.$$ (By § 182, 3.)

But $\frac{mp}{nq}$ is the *product* of the exponent of the quantity and the exponent of the power.

Hence the rule to multiply the exponent of the quantity by the exponent of the power holds for fractional as well as integral exponents.

184. Exercise in performing Operations involving Quantities having Fractional Exponents.

Work mentally the first ten and as many more as possible.

1. Simplify $3\,a^{\frac{2}{3}} \times -5\,a^{\frac{3}{5}}$. *Ans.* $-15\,a^{\frac{19}{15}}$.

2. $9\,a^{\frac{7}{6}}b \times 3\,a^{\frac{5}{6}}b^{\frac{2}{3}}$. 3. $3\,m^{\frac{2}{3}}n^{\frac{4}{5}} \times 6\,m^{\frac{1}{6}}n^{\frac{1}{5}}p$. 4. $a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{5}{6}} \times a^{\frac{1}{2}}bc^{\frac{1}{3}}$.

5. $a^{\frac{m}{n}} \times a^{\frac{p}{q}} \times a^{\frac{r}{s}}$. 6. $12\,x^{\frac{1}{3}}y^{\frac{4}{3}}z^{\frac{1}{2}} \div -2\,x^{\frac{1}{5}}y^{\frac{2}{5}}$. 7. $6\,x^{\frac{a}{b}} \div -2\,x^{\frac{c}{d}}$.

8. $(3\,a^{\frac{1}{2}})^3$. Rᴇᴍᴀʀᴋ. Notice that just as $(ab)^2 = a^2b^2$, so here $(3\,a^{\frac{1}{2}})^3 = 3^3 \times (a^{\frac{1}{2}})^3$. (See § 182).

9. $(9\,a^{\frac{5}{6}})^{\frac{1}{2}}$; $(8\,a^{\frac{6}{7}})^{\frac{1}{3}}$. 10. $(a^{\frac{1}{3}})^{\frac{2}{5}}$; $(-4\,a^{\frac{1}{5}})^2$; $(-27\,a^{\frac{2}{5}})^{\frac{1}{3}}$.

11. $(x^{\frac{3}{4}} - x^{\frac{1}{2}} + 2\,x^{\frac{1}{4}} + 3)(x^{\frac{1}{4}} - 2)$. (§ 39, III.)

12. $(x - y) \div (x^{\frac{1}{3}} - y^{\frac{1}{3}})$; $(a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1)(a^{\frac{1}{3}} - 1)$.

13. $(x^{\frac{1}{4}} - 6\,y^{\frac{1}{3}})(x^{\frac{1}{4}} + 5\,y^{\frac{1}{3}})$. (§ 66.)

14. $(x^{\frac{1}{4}} + y^{\frac{1}{4}})^2$; $(a^{\frac{1}{3}} - 5\,b^{\frac{1}{3}})^3$; $(3\,m^{\frac{1}{5}} - 2\,n^{\frac{2}{5}})^4$.

15. From the sum of $5\,ax^{\frac{1}{2}} - (x+y)^{\frac{1}{2}} + (a-b)^{\frac{1}{3}}$ and $-7\,ax$ $+\,2(x+y)^{\frac{1}{2}} - 3(a-b)^{\frac{1}{3}}$ take $3\,ax^{\frac{1}{2}} + 4(x+y)^{\frac{1}{2}} - 5(a-b)^{\frac{1}{3}}$.

16. Calculate $36^{1\frac{1}{2}} + 4^{3\frac{1}{2}} - 100^{0.5} - 81^{0.75}$.

17. Find the product of $\left(\dfrac{ay}{x}\right)^{\frac{1}{2}}$, $\left(\dfrac{bx}{y^2}\right)^{\frac{1}{3}}$, and $\left(\dfrac{y^2}{a^2b^2}\right)^{\frac{1}{4}}$.

18. Divide $x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.

19. $(a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}})^2 = ?$ $(a^{\frac{3n}{2}} + b^{\frac{3n}{2}}) \div (a^{\frac{n}{2}} + b^{\frac{n}{2}})$.

20. $\sqrt{}(1 + 4\,x^{\frac{1}{3}} - 2\,x^{\frac{2}{3}} - 4\,x + 25\,x^{\frac{4}{3}} - 24\,x^{\frac{5}{3}} + 16\,x^2) = ?$

21. Factor $a^{\frac{3}{2}} - 1$ (§ 70); $a^2 + ab + b^2$ (§ 80, 16).

22. $(x^{\frac{6}{5}} - 9\,x + 33\,x^{\frac{4}{5}} - 63\,x^{\frac{3}{5}} + 66\,x^{\frac{2}{5}} - 36\,x^{\frac{1}{5}} + 8)^{\frac{1}{3}} = ?$

23. Expand $(x^{\frac{2}{5}} - 4)(x^{\frac{2}{5}} + 5)$.

24. Expand $(x^{\frac{2}{3}} + 4\,y^{\frac{2}{3}})(x^{\frac{2}{3}} - 4\,y^{\frac{2}{3}})$.

25. Expand $(x^{\frac{n}{3}} - 2\,y^{\frac{2p}{3}})^3$.

26. Expand $(x^{\frac{1}{2}} - 2\,y^{\frac{1}{2}} - 3\,z^{\frac{1}{2}})^2$.

185. Meaning of Zero and Negative Exponents.

1. Zero exponents arise naturally in division when a letter has the same exponent in both divisor and dividend.

Thus, $x^m \div x^m = x^{m-m} = x^0$. (§ 48.)

But $x^m \div x^m = 1$.

Hence, if the rule for subtracting exponents is to continue to hold, we must take $x^0 = 1$. (§ 57, 8.) We have then this important result:

2. **Theorem concerning 0 Exponent.** *Any finite quantity with exponent* 0 *is equal to* 1.

The meaning of this theorem is understood better if we think of x^0 as being associated with other factors.

For example, $\dfrac{6x^2yz}{x^2uv} = \dfrac{6x^0yz}{uv} = \dfrac{6yz}{uv}$.

The exponent 0 shows that x is used *no* times as a factor of the product, or has dropped out and so does not affect the product of the other factors. x^0 cannot have the value 0, for that would make the whole expression 0, and it is *not* 0.

3. The meaning to attach to negative exponents can be obtained from that for zero exponent. We have

$x^m \times x^{-m} = x^{m+(-m)} = x^0 = 1$. (As just shown).

Or $x^m \times x^{-m} = 1$. (Things equal to same thing are equal to each other.)

Then $x^{-m} = \dfrac{1}{x^m}$ (Division Axiom).

Thus, we are led to take a quantity with a negative exponent to mean the *reciprocal* of the same quantity with a positive exponent.

If $b^{-2} = \dfrac{1}{b^2}$, and $c^{-3} = \dfrac{1}{c^3}$, as just laid down,

$$\frac{ab^{-2}}{c^{-3}d} = \frac{a \times \dfrac{1}{b^2}}{\dfrac{1}{c^3} \times d} = \frac{\dfrac{a}{b^2}}{\dfrac{d}{c^3}} = \frac{ac^3}{b^2d}. \quad (§\ 106.)$$

Here we see b^{-2} in the numerator of the first fraction become b^2 in the denominator of the last, and c^{-3} in the denominator of the first fraction become c^3 in the numerator of the last. In this way we get the following theorem:

4. **Theorem concerning Negative Exponents.** *Any factor can be transferred from the numerator to the denominator or from the denominator to the numerator of a fraction provided the sign of its exponent is changed.*

186. Exercise in Using Zero and Negative Exponents.

Write the values of the following:

1. 10^0. 2. 1000^0. 3. a^0b^0c. 4. $\frac{4x^2y^3}{x^2y}$.

Express the following with positive exponents:

5. $2a^{-5}b$. 6. $m^{-1}n^{-2}$. 7. $5a^{\frac{1}{2}}c^{-\frac{1}{3}}$. 8. $10a^{-\frac{5}{4}}b^{\frac{2}{3}}$.

Write the following fractions without denominators:

9. $\frac{2}{x^{-3}}$. 10. $\frac{1}{a^{-2}b^3}$. 11. $\frac{a^2m^4}{2b^{\frac{1}{3}}}$. 12. $\frac{5}{3^{-1}c^{\frac{2}{3}}d^{-\frac{1}{4}}}$.

(*a*) When no denominator is written, 1, of course, is understood. When all the factors of a numerator are transferred to the denominator (as in Ex. 6), the factor 1 is understood to remain in the numerator

187. Rules giving the Laws for Exponents whether Integral, Fractional, or Negative.

1. *In addition, add the coefficients of similar quantities. The exponents in the literal part remain unchanged.*

2. *In multiplication, add the exponents of the same quantity in the factors for the exponent of this quantity in the product.*

3. *In division, subtract the exponent of a factor in the divisor (denominator) from the exponent of the same factor in the dividend (numerator) for the exponent of this quantity in the quotient.*

4. *In raising to powers, multiply the exponent of a factor quantity by the index of the power to which it is to be raised for the exponent of this quantity in the answer.*

5. *In extracting roots, divide the exponent of a factor quantity by the index of the root for the exponent of this quantity in the answer.*

6. *Any finite quantity with exponent* 0 *is equal to* 1.

7. *Any factor of the numerator or denominator of a fraction can be transferred to the other term of the division by changing the sign of its exponent.*

188. General Exercise in the Simplification of Quantities involving Fractional and Negative Exponents. Answers are to have only positive exponents. Solve as many as possible mentally.

In solving numerical exercises write quantities as powers of prime factors, and then, after simplifying, change all negative exponents to positive ones.

Thus, $8^{-\frac{2}{3}} = (2^3)^{-\frac{2}{3}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}\cdot$

1. $16^{\frac{3}{4}}$.
2. $81^{\frac{1}{4}}$.
3. $100^{-\frac{1}{2}}$.
4. $625^{0.75}$.
5. $x^{\frac{2}{3}} \times x^{-\frac{1}{2}}$.
6. $m^{\frac{2}{3}} \times m^{\frac{3}{2}}$.
7. $(a^{\frac{2}{3}})^{\frac{3}{5}}$.
8. $16^{-\frac{1}{4}}$.
9. $a \times a^{-\frac{2}{5}}$.
10. $(a^4b^{-\frac{2}{3}})^{\frac{3}{2}}$.
11. $100^{-\frac{5}{2}}$.
12. $4\,a^{\frac{1}{3}} \div 2\,a^{-\frac{5}{6}}$.
13. $(\frac{4}{25}\,a^2)^{-\frac{3}{2}}$.
14. $(\frac{27}{125})^{-\frac{2}{3}}$.
15. $a^{\frac{1}{3}} \times a^{-\frac{3}{4}} \times a^{\frac{5}{12}}$.
16. $((a^{-2})^2)^{\frac{3}{4}}$.
17. $(xy)^{x+y} \div x^y y^x$.
18. $(a^{\frac{1}{2}} - a^2)^2$.
19. $(a - b) \div (a^{\frac{1}{2}} - b^{\frac{1}{2}})$.
20. $(x^{\frac{2}{3}} + 2\,x^{-\frac{1}{3}} + 1)^2$.
21. $\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^{-3} \div \left(\frac{ab^{-1}}{a^{-3}b^2}\right)^5$.
22. $\left(\frac{x^{-2}y^2}{x^3y^{-2}}\right)^{-\frac{1}{2}} \times \frac{(y^3x^{-2})^{-1}}{x^3y^{-3}}\cdot$
23. $(a^{\frac{1}{n-1}}a^{\frac{1}{n+1}})^{\frac{1}{n}}$.
24. $8^{-\frac{2}{3}} \div 256^{-\frac{1}{4}}$.
25. $\frac{m^{-1} - n^0}{m^{-2} - n^0m^{-1}} \times m^{-2}$.
26. Square $a^{\frac{1}{2}} + b^{\frac{1}{2}} - c^{\frac{1}{2}}$.
27. $12^0 + 4^{\frac{1}{2}} - 9^{-1} + (-64)^{-\frac{1}{3}} + 27^{\frac{2}{3}}$.
28. $(x^{-\frac{9}{4}} + 2\,x^{-\frac{3}{2}} + 4\,x^{-\frac{3}{4}} + 8)(x^{-\frac{3}{4}} - 2)$.

The table at the top of the next page will be found convenient for reference in simplifying and evaluating radical quantities, and evaluating the roots of quadratic equations.

TABLE OF POWERS AND SQUARE ROOTS

x	x^2	x^3	x^4	x^5	x^6	$x^{\frac{1}{2}}$	x	$x^{\frac{1}{2}}$	x	$x^{\frac{1}{2}}$
1	1	1	1	1	1	1.000	10	3.162	19	4.359
2	4	8	16	32	64	1.414	11	3.317	20	4.472
3	9	27	81	243	729	1.732	12	3.464	21	4.583
4	16	64	256	1024	4096	2.000	13	3.606	22	4.690
5	25	125	625	3125		2.236	14	3.742	23	4.796
6	36	216	1296	7776		2.449	15	3.873	24	4.899
7	49	343	2401			2.646	16	4.000	25	5.000
8	64	512	4096			2.828	17	4.123	26	5.099
9	81	729	6561			3.000	18	4.243	27	5.196

II. RADICAL EXPRESSIONS

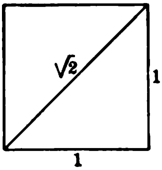
189. Radical Quantities. An indicated root of a quantity is called a **radical quantity**, or a **radical.** Thus, $\sqrt[3]{5}$, $2^{\frac{1}{2}}$ are radical quantities. In $4\sqrt[3]{5a}$, or $4(5a)^{\frac{1}{3}}$, 4 is the **coefficient** of the radical, 3 is the **index** of the root, and $5a$ is the **radicand.**

Radical quantities may be either *rational,* as $\sqrt{25}$, $\sqrt{\frac{4}{9}}$, or *irrational,* as $\sqrt{7}$. A **rational** quantity is the **ratio** of two integral numbers. An **irrational** (or **surd**) quantity cannot be the ratio of two integral numbers.

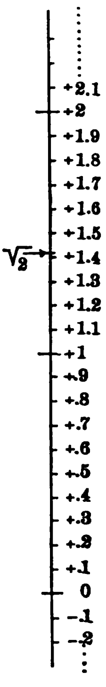
Rational quantities, as $\frac{4}{1}$, $\frac{2}{5}$, $\frac{15}{111}$, give rise, when the numerator is divided by the denominator, to either integers, finite decimals, or repeating decimals, that is, to decimals that *repeat* certain sets of figures. Thus, $\frac{5}{37} = .135135135 \cdots$; $\frac{3}{7} = .428571'428571 \cdots$. When dividing by 7, after ciphers begin to be annexed to the dividend, the possible remainders are 1, 2, 3, 4, 5, 6. When such remainders are all exhausted, the figures of the quotient must repeat in the same order.

Irrational quantities, on the other hand, never give rise to decimals like the preceding classes, but to decimals which do not end and which do not repeat the figures in regular order. An irrational number and unity have no common divisor, that is, are *incommensurable.*

In particular, it can be shown by the continued division process (§ 94) of finding the greatest common divisor that the diagonal of a square and its side, whose ratio equals $\sqrt{2} : 1$, are incommensurable. By the process of § 172, we have $\sqrt{2} = 1.4142^{+}$.



The student can get a fairly good idea of the different kinds of numbers, — positive, negative, integral, rational, and irrational from the following diagram, on which the length $\sqrt{2}$, from 0 to the point indicated by the arrow, is found from the length 1 on this scale by obtaining it as the diagonal of a square whose side is 1. The diagram shows negative numbers as extending *downwards* from 0, and positive numbers *upwards* from 0; it shows also the integral numbers 1 and 2. The decimal division numbers also are examples of rationals, and $\sqrt{2}$ is an example of an irrational number.



To see the difference between rational and irrational numbers, we observe that $\sqrt{2}$ gives rise to a never ending series of decimal figures. As indicated on the diagram, we thus learn, first, that $\sqrt{2}$ lies *between* 1 and 2; next that it lies *between* 1.4 and 1.5. Now, if we subdivided the line from 1.4 to 1.5 into ten equal divisions, $\sqrt{2}$ would lie *between* 1.41 and 1.42; and if we subdivided the line from 1.41 to 1.42 into ten equal divisions, $\sqrt{2}$ would lie *between* 1.414 and 1.415; and if we subdivided the line from 1.414 to 1.415 into ten equal parts, it would lie *between* 1.4142 and 1.4143; and so on indefinitely. Thus, no matter into how many equal decimal divisions the line from 1.4 to 1.5 is divided, $\sqrt{2}$ will always lie *between* two such divisions. It can be shown that the same thing would hold true if the line were divided into any other equal divisions than tenths. The distance from 0 to a point *on* any one of the divisions described would be denoted by a rational quantity. Thus, an irrational number always locates a point different from one located by any rational number.

190. The Duplicate Notation for Radicals. As we have seen, both radical signs and fractional exponents are used to denote roots.

For example:

$$8^{\frac{1}{3}} = \sqrt[3]{8};\ a^{\frac{2}{5}} = \sqrt[5]{a^2};\ 4\,x^{\frac{1}{3}}y^{\frac{2}{3}} = 4\,\sqrt[3]{xy^2}.$$

The reason for the existence and use of both notations is easily explained. The radical sign $\sqrt{\ }$ came into use a century before Newton and Wallis found out that fractional exponents were naturally adapted to denote roots. The use of the radical sign in the meantime had become too firmly intrenched to admit of its displacement. The radical sign notation, it may be remarked, is often shorter than the other. Thus, $(a+b)^{\frac{1}{2}}$ is more troublesome to write than $\sqrt{a+b}$, because both numerator and denominator of the fractional exponent are not written in the radical sign notation.

The use of this double notation, however, is oftentimes quite confusing to the student, and it makes the subject much more difficult than it would otherwise be.

191. Simplification of Radical Quantities. There are four kinds of **reductions** in radicals which we now proceed to investigate.

A radical is not in its simplest form:

1. *If some required root of it can be extracted.*

Thus, $(36\,x^2)^{\frac{1}{2}} = 6\,x;\ 16^{\frac{1}{4}} = 2;\ (27\,m^3n^6)^{\frac{1}{6}} = (3\,mn^2)^{\frac{1}{2}}.$

a. By § 13 we see that taking the square root of the cube root gives the sixth root; the square root of the square root gives the 4th root; the square root of the fourth root gives the 8th root, etc. Thus,

$$(16\,b^4)^{\frac{1}{8}} = [(16\,b^4)^{\frac{1}{4}}]^{\frac{1}{2}} = (2\,b)^{\frac{1}{2}};\ (m^3n^6)^{\frac{1}{9}} = [(m^3n^6)^{\frac{1}{3}}]^{\frac{1}{3}} = (mn^2)^{\frac{1}{3}}.$$

Type form: $$(a^m)^{\frac{1}{mn}} = a^{\frac{1}{n}}.$$

2. *If any factor of the radicand can have the desired root taken of it.*

$$(18)^{\frac{1}{2}} = (9 \times 2)^{\frac{1}{2}} = 3(2)^{\frac{1}{2}}.\quad (\text{§ 182, 1.})$$

$$5(16\,x^5)^{\frac{1}{3}} = 5(8\,x^3 \times 2\,x^2)^{\frac{1}{3}} = 10\,x(2\,x^2)^{\frac{1}{3}}.$$

If the required root of any factor of the radicand can be taken, that factor may be removed from the radicand, and its root used as a factor of the coefficient of the radical.

b. In searching for square factors, test first for 4, then for 9, then for 25, etc.

Thus, $(252)^{\frac{1}{2}} = (4 \times 9 \times 7)^{\frac{1}{2}} = 6(7)^{\frac{1}{2}}$; $450^{\frac{1}{2}} = (9 \times 25 \times 2)^{\frac{1}{2}} = 15(2)^{\frac{1}{2}}$.

Testing for cube factors, look first for 8, then for 27, etc.

Type form: $$(a^n b)^{\frac{1}{n}} = a(b)^{\frac{1}{n}}.$$

3. *If the radicand is fractional.*

To simplify, we multiply both terms of the radicand by a factor such that we can extract the desired root of the resulting denominator.

$$(\tfrac{3}{5})^{\frac{1}{2}} = (\tfrac{3}{5} \times \tfrac{5}{5})^{\frac{1}{2}} \text{ (§ 98) } = (\tfrac{1}{25} \times 15)^{\frac{1}{2}} = \tfrac{1}{5}(15)^{\frac{1}{2}}. \quad \text{(§ 182, 1.)}$$

$$3\left(\frac{5x}{4y}\right)^{\frac{1}{3}} = 3\left(\frac{5x}{4y} \times \frac{2y^2}{2y^2}\right)^{\frac{1}{3}} = 3\left(\frac{1}{2^3y^3} \times 10xy^2\right)^{\frac{1}{3}} = \frac{3}{2y}(10xy^2)^{\frac{1}{3}}.$$

Type form: $$\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{1}{b}(ab^{n-1})^{\frac{1}{n}}.$$

To simplify a quantity whose radicand is fractional, multiply both terms of the radicand by such a quantity as will make the denominator a perfect power of which the required root can be taken. Extract this root and write the result as a factor of the denominator of the coefficient. If possible, simplify the numerator as in 2 above.

4. *If there is an irrational quantity in its denominator.*

Simplifying such a quantity is called "rationalizing its denominator."

Thus, $$\frac{1}{6^{\frac{1}{2}}} = \frac{1}{6^{\frac{1}{2}}} \times \frac{6^{\frac{1}{2}}}{6^{\frac{1}{2}}} = \frac{6^{\frac{1}{2}}}{6}. \qquad (\text{Since } 6^{\frac{1}{2}} \times 6^{\frac{1}{2}} = 6^{\frac{1}{2}+\frac{1}{2}} = 6.)$$

$$\frac{a}{b^{\frac{2}{3}}} = \frac{a}{b^{\frac{2}{3}}} \times \frac{b^{\frac{1}{3}}}{b^{\frac{1}{3}}} = \frac{ab^{\frac{1}{3}}}{b}. \qquad (\text{Since } b^{\frac{2}{3}} \times b^{\frac{1}{3}} = b^{\frac{2}{3}+\frac{1}{3}} = b.)$$

$$\frac{4}{2+3^{\frac{1}{2}}} = \frac{4}{2+3^{\frac{1}{2}}} \times \frac{2-3^{\frac{1}{2}}}{2-3^{\frac{1}{2}}} = \frac{4(2-3^{\frac{1}{2}})}{4-3} = 4(2-3^{\frac{1}{2}}). \quad \text{(§ 65.)}$$

Verify this last result by calculating $\frac{4}{2+3^{\frac{1}{2}}}$ and the answer $4(2-3^{\frac{1}{2}})$ each to three decimal places. Notice by this simplification a long division is saved.

To rationalize a quantity with a monomial denominator, multiply both terms by a monomial which will make the denominator rational.

To rationalize a quantity with a binomial denominator, multiply both terms by the "conjugate surd" to the denominator; that is, the denominator with the sign between its terms changed.

Notice that multiplying by the conjugate surd always makes the radical signs disappear. Why?

192. Exercise in Simplifying Radical Quantities. All problems given in the radical sign notation should be immediately changed into the fractional exponent notation, then solved, and last of all changed back. *Before changing back into the radical sign notation, separate quantities with fractional exponents into two factors, one with integral exponents, and the other with proper fraction exponents.*

Thus, $$5\,a^{\frac{5}{2}}b^{\frac{3}{2}} = 5\,a^2b \times a^{\frac{1}{2}}b^{\frac{1}{2}} = 5\,a^2b\sqrt{ab}.$$

Solve as many as possible of the following mentally:

1. $(25\,a^3b)^{\frac{1}{2}}$.
2. $(27\,a^4b^3c^2)^{\frac{1}{2}}$.
3. $(108\,a^3b)^{\frac{1}{3}}$.
4. $(x^2y^4z^6)^{\frac{1}{2}}$.
5. $(1000)^{\frac{1}{6}}$.
6. $(1\frac{11}{16})^{\frac{1}{2}}$.
7. $\sqrt[3]{a^4x^2y^3}$.
8. $7\sqrt{396\,x}$.
9. $5\sqrt{726}$.
10. $\sqrt{\frac{1}{2}}$.
11. $\sqrt[3]{\frac{1}{2}}$.
12. $\sqrt[3]{\frac{5\,a^2}{9b}}$.
13. $\frac{3}{5}\sqrt{1\frac{2}{3}}$.
14. $\frac{a}{b}\sqrt{\frac{c^2}{d}}$.
15. $\frac{3}{45^{\frac{1}{2}}}$.
16. $\left(\frac{a^{-1}b^{-2}}{x}\right)^{\frac{1}{2}}$.
17. $\sqrt{a^2b^2+a^2c^2}$.
18. $\sqrt[3]{a^{7x}b^{8x}}$.
19. $\sqrt[5]{729\,a^6}$.
20. $\sqrt[6]{64\,a^{18}x^7}$.
21. $\sqrt{(a^2-1)(1+a)}$.

22. $\sqrt[3]{\frac{8}{9}}$. **23.** $\dfrac{3}{3^{\frac{1}{2}}+5^{\frac{1}{2}}}$. **24.** $\dfrac{1}{\sqrt{5}-\sqrt{2}}$.

25. $\sqrt[4]{\dfrac{9\,a^{-2}}{36\,x^{2}}}$. **26.** $\sqrt{\dfrac{a^3}{x^4}-\dfrac{a^5}{x^6}}$. **27.** $\sqrt[10]{\dfrac{32}{x^{15}y^{20}}}$.

28. Find the numerical value of $\sqrt{4\,a^2b}$ when $a=3.51$, $b=16.81$.

a. Notice the result can be had very much more quickly by first simplifying the radical. Check result by calculating radical in original form, noting the difference in the labor of calculation.

29. Calculate $\sqrt{L^2Fm}$ when $L=25.7$, $F=14$, $m=4\frac{4}{7}$.

30. Calculate area of triangle whose sides are 8, 10, and 14 (§ 89), being given from a table (p. 149) that $\sqrt{6}=2.449$.

31. Calculate the diagonal of a rectangular box from one lower corner to opposite upper corner if dimensions are 3, 4, 5 ft. and if it is known from a table (p. 149) that $\sqrt{2}=1.414$.

32. Find u from formula $u=\sqrt{v^2+2\,fs}$ when $v=3$, $f=31\frac{5}{6}$, $s=3$.

33. Find one side of a square equal to the sum of two squares whose sides are respectively 13 m. and 9 m., using the table of square roots.

34. Calculate $\sqrt{8^3\times 6^3\times 72}$, using table of square roots.

SUGGESTION. First write: $8^3=(2^3)^3=2^9$; $6^3=(2\times 3)^3=2^3\times 3^3$; $72=2^3\times 3^2$.

35. Calculate $\sqrt{\dfrac{32^2\times 4^2\times 12^5}{8^4\times 36}}$, using table.

193. Insertion of Coefficients under the Root Sign.

1. $5(3\,a)^{\frac{1}{2}}$.

SOLUTION. $5(3\,a)^{\frac{1}{2}}=25^{\frac{1}{2}}\times(3\,a)^{\frac{1}{2}}=(75\,a)^{\frac{1}{2}}$. (§ 182, 1).

2. $9(2\,m)^{\frac{1}{2}}$. 3. $3\,x(5\,x)^{\frac{1}{3}}$. *Ans.* $(135\,x^4)^{\frac{1}{3}}$.

4. $2\,y(7\,y^3)^{\frac{1}{4}}$. 5. $3\,x^2\sqrt{5\,xy}$. 6. $\frac{1}{2}\sqrt[3]{4}$.

7. $\dfrac{ax}{a-x}\sqrt{\dfrac{a^2-x^2}{a^2x^2}}$. 8. $5\,x(25\,x^2)^{\frac{1}{n}}$. 9. $a^m\sqrt[n]{2\,a^{mn+2}}$.

194. Addition and Subtraction of Radicals. To add radical quantities they are first simplified or reduced to similar radicals.

1. Add $18^{\frac{1}{2}}$, $2\left(\frac{1}{2}\right)^{\frac{1}{2}}$, $5(64)^{\frac{1}{4}}$, and $\frac{3}{8^{\frac{1}{2}}}$.

SOLUTION		CHECKING
$18^{\frac{1}{2}} = 3(2)^{\frac{1}{2}}$	(§ 191, 2.)	$\sqrt{18} = 4.24+$
$2(\frac{1}{2})^{\frac{1}{2}} = 2^{\frac{1}{2}}$	(§ 191, 3.)	$2\sqrt{.50} = 1.41+$
$5(8)^{\frac{1}{2}} = 10(2)^{\frac{1}{2}}$	(§ 191, 1, 2.)	$5\sqrt{8} = 14.14+$
$\frac{3}{8^{\frac{1}{2}}} = \frac{3}{4}(2)^{\frac{1}{2}}$	(§ 191, 4.)	$\frac{3}{\sqrt{8}} = 1.06-.$
Sum $= 14.75(2)^{\frac{1}{2}} = 20.86-.$		$20.85+$

a. By this radical simplification, which is easily and quickly performed, *four* root extractions and one long division are changed into *one* root extraction and one multiplication.

Change quantities written in the radical sign notation to the other before solving. Check some of the solutions, assigning values to the letters partly to test answers and partly for drill in root extraction. Solve as many as possible mentally.

2. $8(125)^{\frac{1}{2}} + 2(80)^{\frac{1}{2}}$.

3. $\sqrt{48\,ab^2} - b\sqrt{75\,a}$.

4. $2(\frac{1}{4})^{\frac{1}{3}} + 8\sqrt[3]{\frac{1}{32}}$.

5. $4\sqrt{128} - 5\sqrt{162} + 16\sqrt{\frac{1}{2}}$.

6. $3\sqrt{75\,a^4b} - 2\,a^2b\sqrt{\frac{12}{b}}$.

7. $3\sqrt[3]{189} - 3\sqrt[3]{875} - 7\sqrt[3]{56}$.

8. $3\,b^2(a^3c)^{\frac{1}{2}} - 2\,ab^3\left(\frac{ac}{b^2}\right)^{\frac{1}{2}}$.

9. $\frac{2}{\sqrt{2}} + \frac{3}{\sqrt{8}} - \frac{1}{\sqrt{18}}$.

10. $2\sqrt{3} - \sqrt{12} + \sqrt[4]{9}$.

11. $\sqrt[3]{\frac{1}{2}} - 2\sqrt[6]{16} + \sqrt[3]{32}$.

12. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{9}} + \sqrt{\frac{1}{8}}$.

13. $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} + \sqrt{\frac{3}{5}}$.

14. $3\sqrt{\frac{5}{2}} + \sqrt{40} + \sqrt{\frac{2}{5}} - \frac{1}{\sqrt{10}}$.

15. $\sqrt[4]{a^2b^8} + \sqrt[3]{a^6b^2} - \sqrt[8]{a^8b^4}$. (§ 31, 10)

16. $\sqrt{2\,ax^2 - 4\,ax + 2\,a} - \sqrt{2\,ax^2 + 4\,ax + 2\,a}$.

17. From $(a-x)(a^2-x^2)^{\frac{1}{2}}$ take $a(a-x)\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$.

18. $(54\,a^{m+6}b^3)^{\frac{1}{3}} - (16\,a^{m-3}b^6)^{\frac{1}{3}} + (2\,a^{4m+9})^{\frac{1}{3}} + (2\,c^3a^m)^{\frac{1}{3}}$.

SUGGESTION. $(54\,a^{m+6}b^3)^{\frac{1}{3}} = (27\,a^6b^3 \times 2\,a^m)^{\frac{1}{3}}$; etc.

19. The areas of three lots in the form of squares lying side by side along a road are 6, 24, and 54 sq. rd., respectively. What is their total frontage on the road? Use table of square roots, page 149.

20. The areas of three squares are 32, 8, and 18 acres respectively. Find the number of rods of fence needed to inclose all of them.

21. One square lot has an area of 75 sq. rd. and another an area of 12 sq. rd. How much more fence will one need than the other?

195. Multiplication and Division of Radicals Having a Common Index. Use the fractional exponent notation in solving.

Type form: $a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$. (See § 182, 1).

1. Multiply $4\sqrt{15}$ by $3\sqrt{35}$.

SOLUTION. $4(5\times 3)^{\frac{1}{2}} \times 3(5\times 7)^{\frac{1}{2}} = 12(5^2\times 21)^{\frac{1}{2}} = 60(21)^{\frac{1}{2}} = 60\sqrt{21}$. *Ans.*
CHECK. $15.49+ \times 17.75+ = 274.9+ = 275^-$.

2. $2(15)^{\frac{1}{2}} \times 3(5)^{\frac{1}{2}}$.

3. $8\sqrt{12} \times 3\sqrt{24}$.

4. $3(8\,a^2)^{\frac{1}{2}} \times (6\,a^3)^{\frac{1}{2}}$.

5. $\sqrt[3]{x+2} \times \sqrt[3]{x-2}$.

6. $4\sqrt{12\,a^3} \div 2\sqrt{3\,a}$.

7. $70\sqrt[3]{9} \div 7\sqrt[3]{18}$.

8. $\sqrt[4]{\frac{b}{a}} \div \sqrt[4]{\frac{a}{b}}$.

9. $\frac{3}{x}\sqrt{\frac{a^2}{x}} \times \frac{4}{3}\sqrt{\frac{x^3}{2\,a^4}}$.

10. $(6^{\frac{1}{2}} + 3^{\frac{1}{2}})(6^{\frac{1}{2}} + 5^{\frac{1}{2}})$.

11. $(9(2)^{\frac{1}{2}} - 7)(9(2)^{\frac{1}{2}} + 7)$.

12. $(7 + 3\sqrt{7})(2\sqrt{7} - 7)$.

13. $(2\sqrt{6} - \sqrt{12} - 3\sqrt{24}) \times 3\sqrt{2}$.

14. $(2\,a + 3\sqrt{x})(3\,a - 2\sqrt{x})$.

15. $(2\sqrt{8} - \sqrt{12} - 5\sqrt{18}) \div 3\sqrt{2}$.

16. $\frac{2}{5}\sqrt{\frac{4}{5}} \times \frac{1}{3}\sqrt{\frac{1}{20}}$.

17. $\frac{3}{8}\sqrt{21} \div \frac{9}{16}\sqrt{\frac{7}{20}}$.

18. $(m + n - \sqrt{mn})(\sqrt{m} + \sqrt{n})$.

19. $(9\sqrt[3]{2} - 6\sqrt[3]{6} - 3\sqrt[3]{8}) \div 3\sqrt[3]{2}$.

20. $(4\sqrt[3]{4} - 3\sqrt[3]{2})(2\sqrt[3]{6} + \sqrt[3]{9})$.

21. $(\sqrt{2} - 3\sqrt{3})^3$.

22. $(5\sqrt{8} + 6\sqrt{12} - 2\sqrt{20})(7\sqrt{2} - 3\sqrt{3} + 4\sqrt{5})$.

Rationalize the denominators of the following, evaluating the first two (see § 191, 4):

23. $\dfrac{2(5)^{\frac{1}{2}}}{5^{\frac{1}{2}}+3^{\frac{1}{2}}}$.

24. $\dfrac{15+14\sqrt{3}}{15-2\sqrt{3}}$.

25. $\dfrac{6-3\sqrt{12}}{2\sqrt{6}+\sqrt{12}}$.

26. $\dfrac{1}{\sqrt{a}+\sqrt{b}}-\dfrac{1}{\sqrt{a}-\sqrt{b}}$.

27. $\dfrac{1}{\sqrt{10}-\sqrt{2}-\sqrt{3}}$.

Suggestion. Make a binomial out of denominator and rationalize (§ 191, 4). Then rationalize the new denominator in a second operation.

28. $\dfrac{\sqrt{2}-\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

29. $\dfrac{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}}$.

30. If $t=2\pi\sqrt{\dfrac{l}{g}}$, and $t'=2\pi\sqrt{\dfrac{l'}{g}}$ show that $t:t'=\sqrt{l}:\sqrt{l'}$.

In the following exercises the results are to be reduced to their simplest form whenever they can be simplified. See § 191.

31. Find the surface of a cube whose volume is 60; whose volume is v.

32. Find the volume of a cube whose surface is 20; whose surface is s.

33. Ratio of area of one square to a second is $\frac{1}{8}$. Find ratio of side of first to side of second. (N. B. *The areas of similar figures are proportional to the squares of the corresponding sides.*)

34. One side of an equilateral triangle is 3. Calculate its altitude, and then find ratio of one half of one side to the altitude. Find same ratio when one side is a. Why are ratios the same?

35. Find the ratio of a side to the diagonal of a square. Let side $= a$.

36. One leg of a right triangle is one half the hypotenuse R. Find the other leg, and then the area.

37. The ratio of one leg of a right triangle to another is 2. Find the ratio of each leg to the hypotenuse.

38. The ratio of one leg of a right triangle to the hypotenuse is $\frac{1}{3}$. Find the ratio of the given leg to the other.

39. Find the area of an equilateral triangle one side of which is 6.

40. Find a side and the area of a square whose diagonal is d.

41. If a regular decagon is inscribed in a circle whose radius is R, one side of the decagon is $\frac{R(\sqrt{5}-1)}{2}$. Find the ratio of the radius to one side.

42. One leg of an isosceles triangle is 6, and the base is $\sqrt{5}$. Find the altitude and area.

43. If an equilateral △ is inscribed in a circle whose radius is R, the perpendicular distance from the center to one side is $\frac{1}{2}R$. Find ratio of a side of the triangle to the radius.

196. Multiplication and Division of Radicals which do not have a Common Index. To perform the multiplication, such quantities must be reduced to a common index. Change quantities written in radical sign notation to fractional exponent notation.

1. Multiply $18^{\frac{1}{3}}$ by $6^{\frac{1}{2}}$.

SOLUTION.

$$18^{\frac{1}{3}} \times 6^{\frac{1}{2}} = (2 \times 3^2)^{\frac{2}{6}} \times (2 \times 3)^{\frac{3}{6}} = (2^2 \times 3^4)^{\frac{1}{6}}(2^3 \times 3^3)^{\frac{1}{6}}. \quad (\S\ 182,\ 2,\ 3.)$$

$$= (2^2 \times 3^4 \times 2^3 \times 3^3)^{\frac{1}{6}} = (2^5 \times 3^7)^{\frac{1}{6}}. \quad (\S\ 182,\ 1.)$$

$$= (3^6 \times 2^5 \times 3)^{\frac{1}{6}} = 3(2^5 \times 3)^{\frac{1}{6}} = 3(96)^{\frac{1}{6}}. \quad \textit{Ans.} \quad (\S\ 191,\ 2.)$$

2. $\sqrt[3]{6} \times \sqrt{2}$.
3. $18^{\frac{1}{2}} \times 6^{\frac{1}{3}}$.
4. $\sqrt[5]{64} \div 2$.
5. $\sqrt[3]{2} \times \sqrt[4]{3}$.
6. $\sqrt[3]{10} \times \sqrt[4]{2}$.
7. $\sqrt[m]{x^p} \div \sqrt[n]{x^q}$.
8. $\sqrt[3]{6} \times \sqrt[6]{\frac{1}{8}} \div \sqrt{8}$.
9. $a\sqrt[m]{x} \times b\sqrt[n]{y}$.
10. $16^{\frac{1}{3}} \times 2^{\frac{1}{2}} \times 32^{\frac{5}{6}}$.
11. $2\sqrt[3]{6} \div 6\sqrt{2}$.
12. $\sqrt[4]{\frac{a}{b}} \times \sqrt[3]{\frac{a^2}{b^2}} \div \sqrt{\frac{b}{a}}$.
13. $\sqrt[3]{4^2} \times \sqrt[9]{8} \times \sqrt[3]{4}$.
14. $(2\sqrt{3} - \sqrt[3]{2})(2 - \sqrt{6})$.
15. $(\sqrt{2} - \sqrt[3]{2})^3$.
16. Arrange in order of magnitude $\sqrt{\frac{4}{3}}$, $\sqrt[3]{\frac{3}{2}}$, $\sqrt[4]{\frac{7}{4}}$.

197. Powers and Roots of Radicals. Change quantities to fractional exponent notation.

1. $[3(ab)^{\frac{1}{2}}]^3$. *Ans.* $27\,ab(ab)^{\frac{1}{2}}$.

2. $(9^{\frac{1}{3}})^2$. 3. $(3 \cdot 3^{\frac{1}{3}})^2$. 4. $(\sqrt{ax^2})^3$.

5. $(2\sqrt{\tfrac{1}{3}})^5$. 6. $\sqrt{\sqrt{a^3}}$. 7. $\sqrt[5]{\sqrt{32 \times 10^5}}$.

8. $(\sqrt[6]{3\,c^2})^3$. 9. $(\sqrt[m]{a^2b})^{m+1}$. 10. $(\sqrt{3}-\sqrt{2})^4$.

11. Simplify $\left(\frac{\sqrt{5}-3}{2}\right)^2 + 3\left(\frac{\sqrt{5}-3}{2}\right) + 1$.

12. Simplify $2\left(\frac{7-\sqrt{17}}{4}\right)^2 - 7\left(\frac{7-\sqrt{17}}{4}\right) + 4$.

13. Verify that $x = 3 + \sqrt{2}$ is a root of equation $x^2 - 6\,x + 7 = 0$.

14. Verify that $\frac{1}{2}(3 \pm 2\sqrt{6})$ are roots of $4\,x^2 - 12\,x = 15$.

15. Verify that $\frac{-11 \pm \sqrt{21}}{10}$ are roots of $5\,y^2 + 11\,y + 5 = 0$.

198. The Square Root of Binomial Surds.

We have, $(\sqrt{2} \pm \sqrt{3})^2 = 2 \pm 2\sqrt{6} + 3 = 5 \pm 2\sqrt{6}$.

To extract the square root of the last expression by the method of § 171, the rational term, 5, must be separated into two parts such that the product of their square roots is $\sqrt{6}$. This can usually be done by inspection when the coefficients are small.

Extract the square root of the following by inspection:

1. $4 + 2\sqrt{3}$.

SOLUTION. $\sqrt{3 + 2\sqrt{3} + 1} = \sqrt{3} + 1$. (§ 171; § 184.)

2. $6 - 2\sqrt{5}$. 3. $9 - 2\sqrt{7}$. 4. $23 - 8\sqrt{7}$.

5. $10 - \sqrt{96}$. 6. $11 + \sqrt{72}$. 7. $28 - 5\sqrt{12}$.

When the coefficients are large, mathematicians have recourse to a formula to solve these problems. Evidently the square root of any binomial surd will have the *form* $\sqrt{x} \pm \sqrt{y}$. It is clear, also, that if the square of $\sqrt{x} + \sqrt{y}$ gives the sum, $a + \sqrt{b}$, the square of $\sqrt{x} - \sqrt{y}$ will give the difference, $a - \sqrt{b}$.

8. Let $a+\sqrt{b}$ or $a-\sqrt{b}$ denote any binomial surd, and $\sqrt{x}+\sqrt{y}$ the square root of the first and $\sqrt{x}-\sqrt{y}$ the square root of the second.

Then, (1) $\sqrt{x}+\sqrt{y}=\sqrt{a+\sqrt{b}}.$

(2) $\sqrt{x}-\sqrt{y}=\sqrt{a-\sqrt{b}}$

whence, $x-y=\sqrt{a^2-b}$ (Mult. Ax., multiplying (1) by (2).)

and $\underline{x+y=a}$ (By squaring eq. (1) and squaring eq. (2), adding, and dividing through by 2.)

$\therefore\ 2x=a+\sqrt{a^2-b}.$ (Add. Ax.)

$2y=a-\sqrt{a^2-b}.$ (Sub. Ax.)

whence $\sqrt{x}=\sqrt{\dfrac{a+\sqrt{a^2-b}}{2}}.$ (Div. and Sq. Root Axs.)

$\sqrt{y}=\sqrt{\dfrac{a-\sqrt{a^2-b}}{2}}.$ (Div. and Sq. Root Axs.)

But $\sqrt{a\pm\sqrt{b}}=\sqrt{x}\pm\sqrt{y}.$ (By hypothesis.)

$$\therefore\ \sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

9. Extract the square root of $37-20\sqrt{3}$, using the formula just found.

Solution. $37-20\sqrt{3}=37-\sqrt{1200}.$ (§ 193.)

Then, $a=37,\ b=1200,$

giving $\sqrt{37-\sqrt{1200}}=\sqrt{\dfrac{37+\sqrt{37^2-1200}}{2}}-\sqrt{\dfrac{37-\sqrt{37^2-1200}}{2}},$

$$=\sqrt{\frac{37+13}{2}}-\sqrt{\frac{37-13}{2}},$$

$$=5-\sqrt{12}=5-2\sqrt{3}.$$

Proof. $(5-2\sqrt{3})^2=25-20\sqrt{3}+4\times 3=37-20\sqrt{3}.$

10. $17+\sqrt{288}.$
11. $56+24\sqrt{5}.$
12. $87-12\sqrt{42}.$
13. $47-4\sqrt{99}.$
14. $56+32\sqrt{3}.$
15. $1+\frac{2}{3}\sqrt{2}.$
16. $x-2\sqrt{x-1}.$
17. $4\frac{1}{3}-\frac{4}{3}\sqrt{3}.$
18. $x+xy-2x\sqrt{y}.$

III. IMAGINARIES

199. Imaginary Quantities. We have seen that such an expression as $\sqrt{-4}$ equals neither $+2$ nor -2, since the square of each of these is $+4$. The same thing is true of the square root of *any* negative number. Because such roots are neither positive nor negative ordinary numbers, they have been called *imaginaries*, though this term is somewhat misleading.

A **pure imaginary** is an even root of a negative number, as $\sqrt{-9}$, $(-6)^{\frac{1}{2}}$. The sum of a real quantity and a pure imaginary, as $2+5\sqrt{-1}$, is called a **complex number**.

The **unit** for ordinary positive numbers is $+1$; that for negative numbers is -1; and that for pure imaginaries is $\sqrt{-1}$, or $(-1)^{\frac{1}{2}}$. The $\sqrt{-1}$ unit is often denoted by the letter i.

1. By the definitions of root and power (§§ 12, 13), we have:

$(\sqrt{-1})^2=-1,$ (Since $\sqrt{-1}$ means a number which
or, $([-1]^{\frac{1}{2}})^2=-1.$ multiplied by itself gives -1.)

$$(\sqrt{-1})^3=(\sqrt{-1})^2(\sqrt{-1})=-1\sqrt{-1},$$

$$(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=-1\times-1=1.$$

Thus, $(\sqrt{-1})^2=-1$; $(\sqrt{-1})^3=-\sqrt{-1}$; $(\sqrt{-1})^4=+1$,
or, $i^2=-1$; $i^3=-i$; $i^4=+1$.

The student is advised to solve some of the following problems as they stand, and then with i substituted for $\sqrt{-1}$, until he gets accustomed to associating the two notations.

2. Add $4+3\sqrt{-1}$ and $7+5\sqrt{-1}$. *Ans.* $11+8\sqrt{-1}$.

3. $(3+2\sqrt{-1})+(4-5\sqrt{-1})$.

4. $(-6-11\sqrt{-1})-(5-3\sqrt{-1})$.

5. $\sqrt{-4}+\sqrt{-49}-2\sqrt{-9}-3\sqrt{-25}$. *Ans.* $-12\sqrt{-1}$.

6. $\sqrt{-9a^4}+5\sqrt{-16a^4}$. (§ 191.) 7. $3\sqrt{-25}-\sqrt{-81}$.

8. $3(-20)^{\frac{1}{2}}-(-80)^{\frac{1}{2}}+6(-45)^{\frac{1}{2}}$. *Ans.* $20(-5)^{\frac{1}{2}}$.

9. $6(-16)^{\frac{1}{2}}-[-5-(-36)^{\frac{1}{2}}]$. 10. $(-a^2+2ab-b^2)^{\frac{1}{2}}$.

11. Multiply $6+2\sqrt{-5}$ by $3-4\sqrt{-5}$.

Solution. In all multiplications, divisions, and raising to powers of imaginary and complex numbers the safe plan is to replace $\sqrt{-1}$ by i. The reason is that following the ordinary rule for multiplication *may mislead.*

Thus, $$(\sqrt{-1})^2=\sqrt{-1}\times\sqrt{-1}=\sqrt{+1}=-1 \textit{ only}.$$

For, by definition of power and root $(\sqrt{-1})^2=-1$.

To change $2\sqrt{-5}$ into the i notation we write $2\sqrt{5}\sqrt{-1}=2\sqrt{5}\,i$. Then, in the answer $18\sqrt{5}\,i$ is changed back to $18\sqrt{-5}$.

For the value of i^2 see 1 of this article.

$$\begin{array}{l}
6+\ \ 2\sqrt{5}\,i \\
\underline{3-\ \ 4\sqrt{5}\,i} \\
18+\ \ 6\sqrt{5}\,i \\
\quad\ \underline{-24\sqrt{5}\,i-8(\sqrt{5})^2i^2\qquad\qquad\qquad} \\
18-18\sqrt{5}\,i-8\times5\times-1=58-18\sqrt{-5}.\quad \textit{Ans.}
\end{array}$$

12. $(5+\sqrt{-3})(7+4\sqrt{-3})$.

13. $(3+\sqrt{-7})(2-\sqrt{-6})$.

14. $(4+(-5)^{\frac{1}{2}})(-6-(-8)^{\frac{1}{2}})$.

15. $(6-2\sqrt{-1})^2$.

16. $\sqrt{-36}\times\sqrt{-25}\times 5$.

17. $3(-6)^{\frac{1}{2}}\times2(-4)^{\frac{1}{2}}\times(-7)^{\frac{1}{2}}$.

18. $(-3+\sqrt{-4})^3$.

19. $6(-3)^{\frac{1}{2}}\div2(-5)^{\frac{1}{2}}$. *Ans.* $\frac{3}{5}(15)^{\frac{1}{2}}$.

20. $2\sqrt{-8}\div\sqrt{-2}$.

21. $2(-32)^{\frac{1}{2}}\div2(-2)^{\frac{1}{2}}$.

22. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$. (See § 191, 4.)

23. $\dfrac{\sqrt{3}-\sqrt{-5}}{\sqrt{3}-\sqrt{-2}}$.

24. Form the product of $x+a$, $x-a$, $x+a\sqrt{-1}$, and $x-a\sqrt{-1}$.

25. Form the product of $y-3$, $y-7$, $y+2+\sqrt{-7}$, and $y+2-\sqrt{-7}$.

26. Verify that both $x=-4+\sqrt{-2}$ and $x=-4-\sqrt{-2}$ satisfy the equation $x^2+8x+18=0$.

27. Verify that $x=6a\pm2a\sqrt{-1}$ satisfy $x^2-12ax+40a^2=0$.

200. Theorems concerning Equations containing Irrationals and Imaginaries.

1. *One irrational quantity cannot equal the sum of a rational quantity and another irrational quantity.*

Let $\sqrt{m}$ and $\sqrt{n}$ denote two different irrationals, m, n, and a being rational quantities, and *suppose* $\sqrt{m} = a + \sqrt{n}$.

Then, $m = a^2 + 2a\sqrt{n} + n,$ (Squaring Axiom.)

whence $-2a\sqrt{n} = a^2 + n - m,$ (Sub. Ax.)

or $\sqrt{n} = \frac{m - n - a^2}{2a}.$ (Div. Ax.)

Thus, the irrational $\sqrt{n}$ equals the rational quantity $\frac{m - n - a^2}{2a}$, which is contrary to § 189. Hence an irrational quantity cannot equal the sum of a rational quantity and another irrational quantity.

2. *In any equation containing rationals and irrationals, the rational part on one side equals the rational part on the other, and the irrational part on one side equals the irrational part on the other.*

Given $a + \sqrt{b} = c + \sqrt{d}$, in which $\sqrt{b}$ and $\sqrt{d}$ are irrationals.

To prove $a = c$, and $\sqrt{b} = \sqrt{d}$.

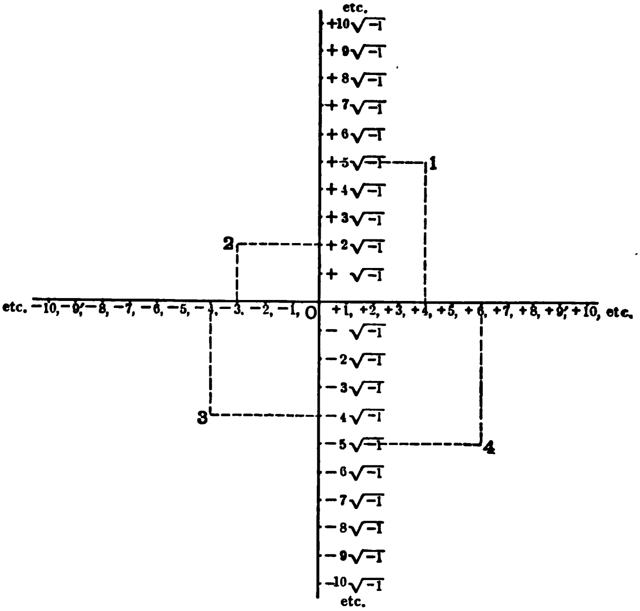
Proof. $\sqrt{b} = c - a + \sqrt{d}.$ (Sub. Ax.)

But this equation cannot exist, by 1 above, so long as $c - a$ is different from 0. Thus, $c - a$ must equal 0, whence $c = a$, and then $\sqrt{b} = \sqrt{d}$. Q.E.D.

The above theorems hold true for real and imaginary quantities as well as for rationals and irrationals.

201. Geometrical Interpretation of Imaginaries or Orthotomic Numbers. Argand's Diagram. If the word imaginary is used to describe the quantities of the preceding article with the idea that *no* real interpretation of them is possible, it is a misnomer.

It was shown in § 40 that -1 is a multiplier or operator which reverses the direction of the multiplicand. Now $\sqrt{-1} \times \sqrt{-1} = -1$. Thus $\sqrt{-1}$ used as a factor *twice* accomplishes as much as -1 used once. Accordingly it appears that $\sqrt{-1}$ can be regarded as a multiplier which turns a multiplicand number through an angle of 90°. Arbitrarily choosing as the positive direction for angles that opposite to the motion of the hands of a clock, we get the accompanying Argand diagram, in which 1, 2, 3, etc., multiplied by $\sqrt{-1}$ give $\sqrt{-1}$, $2\sqrt{-1}$, $3\sqrt{-1}$, etc., respectively; $\sqrt{-1}$, $2\sqrt{-1}$, $3\sqrt{-1}$, etc., multiplied by $\sqrt{-1}$ give -1, -2, -3, etc.; -1, -2, -3, etc., multiplied by $\sqrt{-1}$ give $-\sqrt{-1}$, $-2\sqrt{-1}$, $-3\sqrt{-1}$, etc.; and $-\sqrt{-1}$, $-2\sqrt{-1}$, $-3\sqrt{-1}$, etc., multiplied by $\sqrt{-1}$ give 1, 2, 3, etc., again. The series written vertically containing $\sqrt{-1}$ have been called *orthotomic* numbers instead of imaginaries, because they *cut at right angles* the original algebraic series.



Any ordinary, orthotomic, or complex number (§ 199) represents some point in the plane of the first two. Thus, -3 represents the point three units to the left of the origin; $3\sqrt{-1}$, the point three units directly above the origin; $4+5\sqrt{-1}$ represents the point marked 1 on the diagram, whose abscissa is $+4$ and ordinate $+5$; $-4-4i$, the point marked 3, whose abscissa is -4

and ordinate -4; and so for any number. Evidently the sign conventions are the same as on the graph diagram.

In § 199 complex numbers were added, subtracted, multiplied, divided, and raised to powers. These operations and their results can all be given a meaning on the diagram.

IV. EQUATIONS CONTAINING RADICALS

202. Solution of Equations involving Radicals. Equations involving radicals are solved in a manner different from the solving of ordinary equations. The difference appears in the first part of the solution in which the radicals are eliminated.

1. Solve $2x^{\frac{5}{2}} = 64$, and verify.

SOLUTION. $x^{\frac{5}{2}} = 32.$ (Ax. ?.)

$x^{\frac{1}{2}} = 2.$ (Extracting the fifth root of equals.)

$x = 4.$ (Squaring equals, Power Axiom § 57.)

VERIFICATION. $2(4)^{\frac{5}{2}} = 2 \times 32 \equiv 64.$

2. $3y^{\frac{3}{4}} = 24.$ **3.** $2\sqrt{x} = 16.$ **4.** $5y^{\frac{2}{3}} = 80.$

5. $x^{-\frac{1}{2}} = \frac{1}{5}.$ **6.** $2z^{\frac{5}{4}} = -486.$ **7.** $2x^{-\frac{2}{3}} = \frac{1}{18}.$

8. Solve $\sqrt{x^2 - 1} - x = -1$, and verify.

SOLUTION. $(x^2 - 1)^{\frac{1}{2}} = x - 1.$ (Sub. Ax. The radical *alone* on one side of equation.)

$x^2 - 1 = x^2 - 2x + 1.$ (Squaring Ax. See § 64.)

$2x = 2.$ (Ax. ?.)

$x = 1.$ (Ax. ?.)

VERIFICATION. $\sqrt{1-1} - 1 \equiv -1$, or $0 - 1 \equiv -1.$

a. The plan is to get the *radical* containing x by itself on one side of the equation. Then, by squaring both sides, the radical sign is made to disappear. From thence on the solution is like those with which we are now familiar.

9. $\sqrt{16 + x^2} - x - 2 = 0.$ **10.** $(ax + 2ab)^{\frac{1}{2}} - a = b.$

11. $\sqrt{4x^2 - 6x - 6} = 2x + 4.$ **12.** $1 + 2\sqrt{x} = 7 - \sqrt{x}.$

13. $(x^3 - 9x^2)^{\frac{1}{3}} + 3 = x.$ **14.** $2\sqrt{x} - x = x - 8\sqrt{x}.$

15. $\sqrt{12x-5}-\sqrt{3x-1}=\sqrt{27x-2}.$

SOLUTION

$12x-5-2\sqrt{36x^2-27x+5}+3x-1=27x-2.$ (Power Ax. § 64.)

$\sqrt{36x^2-27x+5}=6x+2.$ (Transposing, and Div. Ax.)

$36x^2-27x+5=36x^2+24x+4.$ (Squaring Ax.)

$\therefore\ x=\frac{1}{51}.$

VERIFICATION. $9\sqrt{-\frac{1}{17}}-4\sqrt{-\frac{1}{17}}=5\sqrt{-\frac{1}{17}}.$

a. The plan in solving such equations is to place *one* radical quantity (preferably the least simple) by itself on one side of the equation, and then to square, thus removing the root sign. Then the remaining radical quantity is placed by itself on one side, and both members are again squared. The most frequent error in such solutions consists in failure to apply theorems I and II in squaring. Notice $\sqrt{12x-5}-\sqrt{3x-1}$ is a binomial, and must be squared as such.

16. $\sqrt{4x+5}-\sqrt{x}=\sqrt{x+3}.$

17. $\sqrt{x+3}+\sqrt{x+8}-\sqrt{4x+21}=0.$

18. $\sqrt{x}+\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}.$

19. $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}}.$

20. Solve $v=\sqrt{\frac{e}{d}}$ for d; also for e.

21. Solve $v=332.4\sqrt{1+0.003665\,t}$ for t.

22. Solve $t:t'=\sqrt{l}:\sqrt{l'}$ for l; also for l'.

23. Solve $s=\sqrt{2-\sqrt{4-q}}$ for q.

24. Solve $t=\sqrt{\frac{2s-2vt}{g}}$ for s; also for v.

25. $\sqrt[3]{64+x^2-8x}=\frac{4+x}{\sqrt[3]{4+x}}.$

26. $\sqrt{\sqrt{x}+3}-\sqrt{\sqrt{x}-3}=\sqrt{2\sqrt{x}}.$

27. $\frac{3x-1}{\sqrt{3x}+1}=1+\frac{\sqrt{3x}-1}{2}.$

CHAPTER VIII

QUADRATIC EQUATIONS*

203. A **quadratic equation** is one which, when reduced to integral form, contains the second but no higher power of the unknown quantity; or, a quadratic equation is an equation of the second degree. (See § 90.)

204. Classification of Quadratics. Determinate † quadratic equations may be classified into two kinds:

(1) Those which contain but one unknown quantity;

(2) Simultaneous quadratics.

Quadratic equations containing but one unknown are either complete or incomplete. A **complete quadratic** contains *both* the second and first powers of the unknown: as, $ax^2 + bx + c = 0$. An **incomplete** (sometimes called **pure**) **quadratic** contains only the second power of the unknown: as, $ax^2 + c = 0$.

In the earlier sections of this chapter we shall take up the more common types of quadratics, and in the last section the more unusual and difficult ones.

I. INCOMPLETE QUADRATICS

205. Solution of Incomplete Quadratics. Incomplete quadratic equations are solved in much the same way as simple equations.

1. Given $\frac{x}{x+1} + \frac{x}{x-1} = \frac{8}{3}$, to find its root and verify.

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* The study of the chapter on logarithms, Chapter X, can be taken up here if desired. It follows naturally the chapter on exponents, since logarithms are exponents.

† For indeterminate equations, see § 297; also, under Diophantus, in § 234.

SOLUTION. $3x^2-3x+3x^2+3x=8x^2-8$. (Mult. Ax.)
$6x^2-8x^2=-8$. (Sub. Ax.)
$-2x^2=-8$. (Simplifying left member.)
$x^2=4$. (Ax. ?)
$x=\pm 2$. (Root Ax. See § 57.)

VERIFICATION. $\frac{2}{2+1}+\frac{2}{2-1}\equiv\frac{8}{3}$. $\frac{-2}{-2+1}+\frac{-2}{-2-1}\equiv\frac{8}{3}$.

a. Nothing is gained by writing $\pm x=\pm 2$, since this equation can give only $x=+2$, or $x=-2$.

b. Notice that the degree of an equation given in the fractional form cannot be told until after it is cleared of fractions. Some equations, apparently like the one just solved, give rise, when cleared, to simple equations, others to incomplete quadratics, others still to complete quadratics.

Solve and verify in the following:

2. $\frac{4x+5}{7x-1}=\frac{x+2}{5x-3}$.
3. $\frac{x+2}{x-2}+\frac{x-2}{x+2}=\frac{26}{x^2-4}$.
4. $\frac{5}{4x}-\frac{13}{8x}=-\frac{2}{3}x$.
5. $\frac{1+x}{1-x}=\frac{x+25}{x-25}$.
6. $ax^2+b=bx^2+a$.
7. $a^2x^2-b^2+c^2x^2=0$ (§ 158, 28).
8. $\sqrt{4x^2-1}=6x-4\sqrt{4x^2-1}$.
9. Solve $2a^2+2b^2=4m^2+c^2$ for m.

Solve as many as possible of the following mentally, Exs. 10–13 for x, and 14–19 for the letter named:

10. $13x^2-19=7x^2+5$.
11. $(3x-7)(3x+7)=32$.
12. $x^2-3=\frac{4x^2+18}{9}$.
13. $mx^2=a^2-nx^2$.
14. $a^2=b^2+c^2-2cp$ for b.
15. $d=\sqrt{a^2+b^2+c^2}$ for c.
16. $A=\pi(R^2-r^2)$ for r.
17. $s=\frac{1}{2}gt^2$ for t.
18. $V=\pi r^2a$ for r.
19. $H=E^2t/R$ for E.

206. Problems involving the Solution of Incomplete Quadratics.

1. Find the area correct to one decimal place of a circle whose area is 243 sq. in., from the formula $a=\pi r^2$, using 3.1416 for π.

2. Find correct to two decimal places one side of a square whose diagonal is 72 m.

3. Find the altitude of an equilateral triangle one side of which is 6 ft.

4. Find the radius of a sphere whose surface is 600 sq. in., from the formula $s = 4\pi r^2$.

5. Find to two decimal places the radius of the base of a cylinder whose volume is 231 cu. in. and whose altitude is 6 in., from the formula $v = \pi a r^2$.

6. Find to two decimal places the radius of the base of a cone whose volume is 251 cu. cm. and whose altitude a is 22 cm., from the formula $v = \frac{1}{3}\pi a r^2$.

7. The area of a sector of a circle is 25 sq. ft. and its angle at the center is 20°. Find the radius from the formula $a = \frac{20^\circ}{360^\circ}\pi r^2$.

II. COMPLETE QUADRATICS

207. Solution of Complete Quadratic Equations. There are three distinct methods of solving complete quadratics with which the student is asked to familiarize himself, viz.: (1) **By factoring**; (2) **By completing the square**; (3) **By use of a formula.**

Of these the factoring method is not practical when irrationals appear in the roots; otherwise, it generally gives the quickest and easiest solution. Review § 80, 1–4.

208. Solution of Complete Quadratics by Factoring. (See § 90.)

1. Solve and verify in the equation $x^2 - 13x = 48$.

SOLUTION. $x^2 - 13x - 48 = 0$. (Ax. ? Right member made 0.)

$(x - 16)(x + 3) = 0$. (Factoring.)

$$\left.\begin{aligned} \therefore x - 16 &= 0, \text{ whence } x = 16 \textit{ Ans.} \\ x + 3 &= 0, \text{ whence } x = -3 \textit{ Ans.} \end{aligned}\right\} \text{(Setting each factor equal to 0.)}$$

VERIFICATION. $16^2 - 13 \times 16 \equiv 48$; $(-3)^2 - 13(-3) \equiv 48$.

2. $x^2 - 11x = 42$.

3. $x^2 - 12x + 35 = 0$.

4. $x^2 - 15x = 0$.

5. $2x^2 - 11x + 12 = 0$.

6. $6x^2 - x = 15$. 7. $4x^2 + 7x + 3 = 0$.

8. $\frac{x^2}{10} + 350 - 12x = 0$. 9. $96x^2 = 4x + 15$.

10. $x^2 - (a + b)x + ab = 0$. 11. $\sqrt{4v + 17} = 4 - \sqrt{v + 1}$.

12. Separate the number 266 into two factors whose difference is 5.

13. Find three consecutive numbers whose sum is equal to the product of the first two.

14. A man traveled 60 mi., and if he had traveled 3 mi. more per hour, he would have required 1 hr. less to perform the journey. At what rate did he travel?

a. Algebra is a *formal* science, built up without reference to particular problems. When it is used to solve problems which by their nature do not admit of negative answers, such answers are discarded. Thus, -15 in Ex. 14 is ignored. Only in problems which admit of negatives are negative answers retained.

15. The perimeter of a field is 360 rd. and its area is 50 acres; find its length and breadth.

16. A and B start at the same time from the same place. After $2\frac{1}{2}$ hr., A, who covers 1 km. in 3 min. less time than B, has traveled 2.5 km. more than B. How many minutes does each need to travel a kilometer?

209. Construction of Equations, their Roots being given. Converse operation to that of preceding article.

1. Construct the equation whose roots are 4 and -2.

SOLUTION. $[x-4][x-(-2)] = [x-4][x+2] = x^2 - 2x - 8 = 0$.
Hence $x^2 - 2x - 8 = 0$ is the equation sought.

PROOF. $x^2 - 2x - 8 = (x-4)(x+2) = 0$. (Factoring.)
Then, § 208, $x - 4 = 0$, whence $x = 4$; and $x + 2 = 0$, whence $x = -2$.

Form the equations whose roots are as follows:

2. $2, -1$. 3. $3, 7$. 4. $-2, 6$. 5. $-3, -8$. 6. a and b.
7. $\sqrt{2}$ and $-\sqrt{2}$. 8. 25 and -70. 9. $-20a$ and $30a$.

10. Make a rule to construct an equation, being given its roots.

11. Construct the equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

SOLUTION

$$[x-(2+\sqrt{3})][x-(2-\sqrt{3})]=[x-2-\sqrt{3}][x-2+\sqrt{3}]=0,$$

or, $x^2-4x+1=0$. *Ans.* (See § 195.)

12. Construct the equation whose roots are $-3-\sqrt{7}$ and $-3+\sqrt{-7}$.

210. Solution of Quadratic Equations by "Completing the Square." (See § 91.) To solve an equation by this method, it is *first* reduced to the form $a^2x^2+bx=c$, if it is not already in this form.

The *second* step consists in completing the square in the left member. We regard the two terms of the equation which contain x^2 and x as corresponding to a^2+2ab in $a^2+2ab+b^2$. The problem is to find b^2 from the other two terms and add it to them.

The third term is found from formula as follows:

$$(2ab\div 2\sqrt{a^2})^2=b^2, \text{ or } \left[\frac{\text{2d term}}{2\times\text{square root of 1st term}}\right]^2.$$

Solve the following equations by completing the square, and verify:

1. $3x^2-10x+8=0$.

SOLUTION

$3x^2-10x=-8$. (Sub. Ax. Known term to right member.)

$\therefore 9x^2-30x=-24$. (Mult. Ax. Coefficient of x^2 made a positive perfect square.)

$\left[\frac{-30x}{2\sqrt{9x^2}}\right]^2=25$. (The second term is divided by twice the square root of the first term, and the quotient is squared.)

$\therefore 9x^2-30x+25=1$. (Add. Ax. Result just found, 25, is added to both members; $25-24=1$.)

$\therefore 3x-5=\pm 1$. (Root Ax. The square roots of equals are equal. See § 205, *a*.)

$\therefore 3x=5+1=6$, or $5-1=4$. (Sub. Ax.)

$\therefore x=2$, or $\frac{4}{3}$. *Ans.* (Div. Ax.)

VERIFICATION. $3\times 2^2-10\times 2+8\equiv 0$. $3(\frac{4}{3})^2-10\times\frac{4}{3}+8\equiv 0$.

2. $x^2-10x-56=0$.
3. $x^2-6x=135$.
4. $x^2-28x=60$.
5. $x^2-18x-63=0$.
6. $x^2-0.4x=23$.
7. $x^2-0.04x=1.254$.
8. $x^2-11x=-28$.

a. Fractions can always be avoided by first multiplying the equation through by four times the coefficient of x^2, *merely indicating the multiplications in the left member.*

This is called the ***Hindu Rule*** for solving quadratics. Notice that the quantity squared and added to complete the square by this rule is always the coefficient of x in the given equation. It is not always necessary to multiply by so large a multiplier, or, often, any multiplier at all, to avoid fractions. Thus, this was not necessary in Exs. 2–7 above.

b. A simple rule which will solve any quadratic directs us first to *divide the equation through by the coefficient of* x^2. *Then the quantity added to complete the square is always the square of half the coefficient of* x. Explain why.

An objection to this rule is that it often gives rise to large fractions, and gives the student no chance to exercise his judgment, before he begins, as to what is the wisest thing to do, whether to multiply or divide through, and by what quantity, so as to shorten most the subsequent calculations, and avoid error. What is best to do in any given problem is learned from experience.

In the next four problems, following the Hindu Rule is the best that can be done, if fractions are to be avoided. Solve them by the rules in *a* and *b*, and compare the solutions.

9. $2x^2+3x=27$.
10. $3x^2-13x=-10$.
11. $5x^2-7x=-2$.
12. $2x^2-5x=42$.
13. $7x^2+2x=32$.
14. $8x^2-22x=21$.
15. $84x^2+45=129x$.
16. $3x+4=39x^{-1}$.
17. $\frac{19}{5}x=\frac{11}{6}-x^2$.
18. $\dfrac{x+22}{3}-\dfrac{4}{x}=\dfrac{9x-6}{2}$.
19. $\dfrac{m+3}{m-2}-\dfrac{24}{m}=-2$.
20. $\dfrac{2t^2-4t-3}{2t^2-2t+3}=\dfrac{t^2-4t+2}{t^2-3t-2}$.
21. $\sqrt{5n+11}=\sqrt{3n+1}+2$.
22. $\sqrt{2z}+2\sqrt{2z+5}=2\sqrt{6z+4}$.

For the verification in the following eight problems see § 197, Ex. 13, and § 199, Ex. 27.

23. $x^2-4x+2=0$.
24. $m^2+5m+5=0$.

25. $(x-1)(x-2)=1.$

26. $8x^2-20x=21.$

27. $x+\frac{1}{x}=\frac{4}{\sqrt{3}}.$

28. $x^2-6x+10=0.$

29. $\frac{1}{x+\frac{1}{x}}=1.$

30. $7(x+7)+\frac{7(3x+50)}{x}=0.$

31. $x^2-2ax-3a^2=0.$

32. $y^2-4by-5b^2=0.$

33. $ax^2+bx+c=0.$

Solution. $4a^2x^2+4abx=-4ac.$ (Hindu Rule, § 210, *a*.)

$4a^2x^2+4abx+b^2=b^2-4ac.$ (Completing the square.)

$2ax+b=\pm\sqrt{b^2-4ac}$ (Sq. root Axiom.)

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$ (Sub. and Div. Axs.)

Verification.

$$a\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)^2+b\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)+c\doteq 0.$$

$$\frac{+b^2-2b\sqrt{b^2-4ac}+b^2-4ac}{4a}+\frac{-b^2+b\sqrt{b^2-4ac}}{2a}+c=0.$$

$$(2b^2-2b\sqrt{b^2-4ac}-4ac)+(-2b^2+2b\sqrt{b^2-4ac})+4ac\equiv 0.$$

34. $mx^2-2nx+p=0.$

35. $x^2-(a+b)x+ab=0.$

36. $x^2+(n+1)x+n=0.$

37. $y^2-(4a+10)y+40a=0.$

38. $9a^4b^4z^2-6a^3b^3z=b^2.$

39. $(m-n)x^2-nx=m.$

40. $bx^2-ax+a+cx^2=b-cx.$

Suggestion. In all such problems the terms containing x^2 must be combined into one term by factoring, and also the terms containing x, and the known terms are to be transposed to the right member, so as to reduce the equation to the form $mx^2+nx=p$.

41. $t^2+3bt=5ct+15bc.$

42. $x^2-3bx=2ax-6ab.$

43. $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{a-b}.$

44. $\frac{my}{a-y}+m=\frac{n(y+2m)}{m+n}.$

45. Make a rule for the solution of quadratic equations by completing the square covering all the different steps. Include literal equations.

46. $s+\sqrt{1-s^2}=1.2.$

47. $2s-\sqrt{1-s^2}=.25.$

48. $t+\frac{1}{t}=24.$

49. $2c+\sqrt{1+c^2}=17.$

In Ex. 50 solve for s and in Ex. 51 for c.

50. $2s+\sqrt{s^2-1}=m$. 51. $c+\sqrt{1-c^2}=p$.

52. The diagonals of a rectangle are $2x^2+x$ and $x^2+13x-35$. Find both values of x and the diagonals.

53. If a train had traveled 5 mi. an hour faster, it would have needed one hour less to cover 150 mi. Find the rate of the train.

54. The perimeters of two similar polygons are expressed by $2x^2+3x$ and $3x+5$ and the two corresponding sides by 3 and 5. Find values of x first and then find the perimeters.

55. Two launches race over a course of 12 mi. The first goes $7\frac{1}{2}$ mi. an hour. The other, having a start of 10 min., passes over the first half of the course with a certain speed, but increases its speed over the second half by 2 mi. per hour, winning the race by 1 min. What is the speed of the second launch in the first half of the course?

211. Solution of Quadratic Equations by Means of a Formula. Solve the quadratic $ax^2+bx+c=0$, regarding it as a type of all quadratics, getting (§ 210, Ex. 33)

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

To solve any particular quadratic, we reduce it to the form $ax^2+bx+c=0$, if not already in this form, and then substitute the values of a, b, and c from the given problem in the formula just found.

The pupil should solve the equation $ax^2+bx+c=0$ repeatedly, until he becomes familiar with the process and remembers the answer, and can solve the equation mentally, writing down the answer formula.

1. Solve $3x^2-7x=6$ and verify.

SOLUTION. Writing the given equation underneath the type form we have

$$ax^2+bx+c=0,$$
$$3x^2-7x-6=0.$$

We see now that in this particular problem $a = 3$, $b = -7$, $c = -6$. Substituting these values for a, b, and c in

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

we get $$x = \frac{-(-7) \pm \sqrt{(-7)^2 - (4 \times 3 \times -6)}}{2 \times 3},$$

or $$x = \frac{7 \pm \sqrt{49 + 72}}{6} = \frac{7 \pm 11}{6} = 3, \text{ or } -\frac{2}{3}.$$

VERIFICATION. $3(3)^2 - 7 \times 3 \equiv 6.$ $3(-\frac{2}{3})^2 - 7 \times (-\frac{2}{3}) \equiv 6.$

2. $9x^2 + 18x + 8.$
3. $7r^2 + 20r + 12 = 0.$
4. $12x^2 - 11x + 2 = 0.$
5. $6x^2 + 17x = 3.$
6. $x^2 + (n+1)x + n = 0.$
7. $5x^2 + 9x + 12 = 4x^2 + x.$
8. $12 - 51x = 36 + 6x^2.$
9. $10 + 15T + T^2 = 26T.$
10. $(m-n)x^2 - nx = m.$
11. $y^2 - 1 = 4ay - a^2.$
12. $\frac{4x}{x-1} - \frac{x+3}{x} = 4.$
13. $\frac{x}{9(x-1)} = \frac{x-2}{6}.$
14. $\frac{x}{a} + \frac{a}{x} = \frac{2}{a}.$
15. $\sqrt{a+x} - \sqrt{2x} = \frac{2a}{\sqrt{a+x}}.$

16. Find two consecutive numbers the sum of whose reciprocals is $\frac{9}{20}$.

17. A rectangular park 56 rods long and 16 rods wide was surrounded by a street of uniform width containing 4 acres. What was the width of the street?

18. The perimeter of a field is a rods and it contains b acres. What are its dimensions?

19. A merchant sold a coat for $11, and gained a number of per cent equal to the number of dollars the coat cost him. What was his per cent of gain?

20. A boatman rowed 8 miles up a stream and back in 3 hours. If the velocity of the current was 2 miles an hour, what was his rate of rowing in still water?

III. HIGHER EQUATIONS CONTAINING ONE UNKNOWN AND QUADRATICS

212. Higher Equations solved as Quadratics. By *higher* equations are meant equations of a degree higher than the second. (See § 203.)

1. Solve $x^4 - 5x^2 = -4$.

Solution. $4x^4 - 20x^2 = -16$. (Hindu Rule.)

$4x^4 - 20x^2 + 25 = 9$. (Completing Square, Ax. ?)

$2x^2 - 5 = \pm 3$. (Ax. ?)

$x^2 = 4$, or 1. (Ax. ?)

$x = \pm 2$ or ± 1. (Ax. ?)

Verification. $(\pm 2)^4 - 5(\pm 2)^2 \equiv -4$. $(\pm 1)^4 - 5(\pm 1)^2 \equiv -4$.

2. $x^4 - 13x^2 + 36$.

3. $x^4 - 18x^2 + 32 = 0$.

4. $z^4 - 4z^2 = 45$.

5. $9u^4 + 5u^2 = 4$.

6. $t^4 - 24t^2 = 81$.

7. $7y^4 + y^2 = 350$.

8. $x^6 - 7x^3 = 8$.

Suggestion. Solve first for x^3 by completing the square, and then for x, getting two values for x.

9. $y^6 + 37y^3 = 1728$.

10. $m^6 - 19m^3 = 216$.

11. $x^2 + \dfrac{1}{x^2} = \dfrac{17}{4}$.

12. $\dfrac{x^2 - 1}{9} + \dfrac{1}{x^2} = 1$.

13. $16\left(x^2 + \dfrac{1}{x^2}\right) = 257$.

14. $x^2 - 4 + \dfrac{72}{x^2 - 8} = 21$.

15. $x^2 + \dfrac{a^2b^2}{x^2} = a^2 + b^2$.

16. $(x^2 + 2)^2 + 198 = 29(x^2 + 2)$.

213. Construction of Higher Equations.

In § 208 we saw how to solve quadratics by the factoring method, and in § 209 how to construct an equation when its roots are given. Equations of higher degree are solved and constructed in a similar manner.

1. Construct the equation whose roots are 3, 4, and -5.

Solution. $[x - 3][x - 4][x - (-5)] = x^3 - 2x^2 - 23x + 60 = 0$. *Ans.*

Form the equations whose roots are as follows:

2. $2, 1, -3.$ 3. $1, 2, 4.$ 4. $-5, 4, 0.$ 5. $-6, -7, 1.$

6. $1, 2, 3, 2.$ 7. $1, \frac{1}{2}, \frac{1}{3}, 2.$ 8. $a, b, c.$ 9. $a, -b, c, -d.$

10. Make a rule to construct an equation of any degree when its roots are given.

11. $1, -1, 2, -2, 3.$ 12. $\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, 0.$

13. $2 \pm \sqrt{3}, 1, -3.$ 14. $1, -3, 3 \pm \sqrt{-1}.$

15. $2 \pm \sqrt{-2}, 1 \pm \sqrt{-1}.$ 16. $\pm\sqrt{3}, 2 \pm \sqrt{-3}, 0.$

214. Number of Roots in an Equation. The preceding article makes it clear that the number of roots in an equation equals the degree of the equation, but some of the roots are very often imaginary.

215. Solution of Equations of Higher Degrees by Factoring. Reverse operation. Before beginning this article the pupil should review §§ 83–85 and 203.

1. Solve $x^3 = a^3$ and verify.

SOLUTION. $x^3 - a^3 = (x-a)(x^2+ax+a^2) = 0.$ (§ 79.)

$$\left.\begin{aligned} x - a = 0,\ x = a. \quad \textit{Ans.} \\ x^2 + ax + a^2 = 0, \end{aligned}\right\} \text{(Setting each factor equal to 0.)}$$

$$4x^2 + 4ax = -4a^2, \qquad \text{(Hindu rule.)}$$

$$4x^2 + 4ax + a^2 = -3a^2. \qquad \text{(Completing square, Ax. ?)}$$

$$2x + a = \pm a\sqrt{-3}. \qquad \text{(Ax. ?)}$$

$$x = \frac{-a \pm a\sqrt{-3}}{2}. \quad \textit{Ans.} \qquad \text{(Ax. ?)}$$

VERIFICATION. $a^3 = a^3$; $\left(\frac{-a \pm a\sqrt{-3}}{2}\right)^3 = a^3.$

The solution shows that there are *three* cube roots of a^3 instead of one.

Find all the roots of the following by factoring method, setting each prime factor equal to zero, and solving prime factors of the second degree equal to zero by the method of § 210:

2. $x^3 = 8.$ 3. $x^3 + 64 = 0.$ 4. $x^4 = a^4.$

5. $x^6 - 1 = 0.$ 6. $216 + 125x^3 = 0.$ 7. $x^6 - 64 = 0.$

8. $6x^3+17x^2+12x=0.$ 9. $x^3+x^2+x+1=0.$ (§ 81, 1.)

10. $x^3+4x^2-4x-16=0.$ 11. $x^4-2x^3+32x=16x^2.$

12. $x^4-13x^2+40=0.$ 13. $y^3+3y^2+3y+1=0.$

14. $x^4-16=0.$ 15. $x^3-19x-30=0.$ (§ 80, 20.)

16. $x^3-7x+6=0.$ 17. $12x^3-28x^2+17x-3=0.$

18. $2x^4-3x^3-4x^2+3x+2=0.$

19. $4x^3-12x^2+9x-1=0.$

20. $6x^4+25x^3+5x^2-60x-36=0.$

21. $x^4+a^4=0.$ Suggestion. Factor as in § 79, II, and treat $\sqrt{2}$ as if it were a rational quantity, solving by § 210.

22. $x^4+64=0.$ 23. $4x^4+81=0.$ 24. $4y^4+625=0.$

From § 213 it is clear that if we have any means of knowing any of the roots of an equation, we can remove these roots from the equation by dividing its left member (right member having been made 0) by the binomial differences between x and such roots.

25. $x^3+4x^2+6x-11=0.$

Solution. If this equation has a rational root it is either 1 or 11. Substituting $x=1$ in the equation we see it is satisfied. Hence $x-1$ is a factor of the left member by the divisibility theorem, § 69.

Dividing $x^3+4x^2+6x-11$ by $x-1$ we get $x^2+5x+11$.

Thus, $x^2+5x+11=0$ is the equation containing the remaining roots. Solve this equation.

26. $x^4+2x^3-x=30.$ 27. $x^4-6x^3+27x=10.$

IV. SIMULTANEOUS QUADRATICS

216. Simultaneous Quadratics. Degree of Simultaneous Equations. The **degree** of an integral equation is the greatest exponent, or sum of exponents, of the unknown or unknowns in any one term.

Notice that $xy=5$ is a quadratic, as well as $x^2+y^2=16$. We may think of x^2, xy, y^2 as denoting *areas*, while x and y terms denote lengths only.

217. Classes of Simultaneous Quadratics. The variety of problems and solutions in simultaneous quadratics is large. We shall next study examples of three or four of the most common cases in simultaneous quadratic equations, and then in the last section of this chapter examples of the more difficult kinds of problems.

218. Simultaneous Equations, one of which is Quadratic and the other Simple, can always be solved by *substitution* method (§ 143).

1. Given $\begin{cases}(1)\ 3x^2+2y^2-y=48,\\(2)\ 2x-y+3=0.\end{cases}$ to find x and y.

SOLUTION. (2_1) $2x=y-3$; (2_2) $x=\frac{y-3}{2}$. (Axs. ?)

(1) $3\left(\frac{y-3}{2}\right)^2+2y^2-y=48.$ (§ 143.)

$\frac{3y^2-18y+27}{4}+2y^2-y=48.$

$3y^2-18y+27+8y^2-4y=192.$ (Ax. ?)

$11y^2-22y-165=0.$ (Ax. ?)

$y^2-2y-15=0.$ (Ax. ?)

$(y-5)(y+3)=0.$ (§ 208.)

$\therefore y=5.$ *Ans.*, Or $y=-3.$ *Ans.*

(2_2) $x=\frac{5-3}{2}=1.$ *Ans.* (2_2) $x=\frac{-3-3}{2}=-3.$ *Ans.*

VERIFICATION. (1) $3\times1^2+2\times5^2-5\equiv48.$

$3(-3)^2+2(-3)^2-(-3)\equiv48.$

2. $\begin{cases}x^2+2y^2=34,\\x+y=7.\end{cases}$

3. $\begin{cases}x+4y=23,\\x^2+3xy=54.\end{cases}$

4. $\begin{cases}3x-y=11,\\3x^2-y^2=47.\end{cases}$

5. $\begin{cases}3x(y+1)=12,\\3x=2y.\end{cases}$

6. $\begin{cases}x^2+xy+y^2=343,\\2x-y=21.\end{cases}$

7. $\begin{cases}2m^2-3mn+n^2=14,\\2m-n=7.\end{cases}$

8. $\begin{cases}2u-3v=11,\\\frac{4}{u}-\frac{3}{v}=-\frac{17}{7}.\end{cases}$

9. $\begin{cases}u+\frac{1}{3}=\frac{2u+v}{3},\\\frac{u+v}{u}=\frac{4u-v}{2}.\end{cases}$

10. $\begin{cases}ax+by=p,\\cx^2+dy^2=q.\end{cases}$

11. $\begin{cases}s=ut+\frac{1}{2}gt^2,\\v=u+gt.\end{cases}$ u and t unknowns.

219. Simultaneous Quadratic Equations solved by Squaring, Adding or Subtracting, and Extracting the Square Root. — This method is applicable only for certain combinations of the given coefficients.

1. Given $\begin{cases} (1) \quad 9x^2 + 4y^2 = 394, \\ (2) \quad 3x - 2y = 2, \end{cases}$ to find x and y.

SOLUTION

$$\begin{array}{llllll}
(1) & 9x^2 & & +4y^2 & = 394 & \\
(2_1) & 9x^2 & -12xy & +4y^2 & = 4 & (\text{Ax. ?}) \\
\hline
(3) & & 12xy & & = 390 & (\text{Ax. ?}) \\
(1) & 9x^2 & & +4y^2 & = 394 & \\
\hline
(4) & 9x^2 & +12xy & +4y^2 & = 784 & (\text{Ax. ?}) \\
(4_1) & 3x & +2y & & = \pm 28 & (\text{Ax. ?}) \\
(2) & 3x & -2y & & = 2 & \\
\hline
 & 6x & & & = 30, \text{ or } -26 & (\text{Ax. ?}) \\
 & x & & & = 5, \text{ or } -\tfrac{13}{3}. \quad \textit{Ans.} & (\text{Ax. ?}) \\
 & & 4y & & = 26, \text{ or } -30. & (\text{Ax. ?}) \\
 & & y & & = 6\tfrac{1}{2}, \text{ or } -7\tfrac{1}{2}. \quad \textit{Ans.} & (\text{Ax. ?})
\end{array}$$

VERIFICATION. (1) $9(5)^2+4(6.5)^2 \equiv 394$, or (2) $9(-\tfrac{13}{3})^2+4(-\tfrac{15}{2})^2 \equiv 394$.

a. The general plan of this method is to calculate the value of the sum, or difference, or both of two quantities by processes like that just given. Thus, in the preceding problem we *had given* the value of $3x - 2y$, and we proceeded to calculate the value of $3x + 2y$. From these two it was easy to calculate x and y themselves. In Ex. 2, calculate the value of $x - y$.

2. $\begin{cases} x^2 + y^2 = 178, \\ x + y = 16. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 89, \\ xy = 40. \end{cases}$

4. $\begin{cases} xy = 518, \\ x + y = 51. \end{cases}$

5. $\begin{cases} x+y=9, \\ x^2+xy+y^2=61. \end{cases}$

6. $\begin{cases} 2x + y = 7, \\ 4x^2 + y^2 = 25. \end{cases}$

7. $\begin{cases} 2x + 5y = 19, \\ 4x^2+25y^2=241. \end{cases}$

8. $\begin{cases} x^3 + y^3 = 243, \\ x + y = 9. \end{cases}$

SUGGESTION. Begin by dividing Eq. (1) by (2), getting (3). Then use (3) and (2).

9. $\begin{cases} x^3 - y^3 = 152, \\ x - y = 2. \end{cases}$

10. $\begin{cases} x^3 + y^3 = 37, \\ x^2 - xy + y^2 = 37. \end{cases}$

11. $\begin{cases} x^4 + x^2y^2 + y^4 = 21, \\ x^2 + xy + y^2 = 7. \end{cases}$

12. $\begin{cases} x^2 + xy = 77, \\ xy + y^2 = 44. \end{cases}$

220. Problems involving the Solution of Simultaneous Quadratics.

1. Find two numbers which multiplied give 576 and divided the one by the other give $2\frac{1}{4}$.

2. The diagonal of a rectangular box is 125 in., the area of the lid is 4500 sq. in., and the sum of three conterminous edges is 215 in. Find the lengths of the edges.

3. The floor of a certain room has 270 sq. ft., each of the two side walls 153 sq. ft., and each of the two end walls $127\frac{1}{2}$ sq. ft. Find the dimensions of the room.

4. The front wheel of a carriage makes 6 revolutions more than the rear wheel in running 36 yd.; if the circumference of each wheel were 1 yd. more, the front wheel would make only 3 revolutions more than the rear wheel in running the same distance. Find the circumference of each wheel.

Suggestion. Eliminate xy between (1) and (2). Then follow § 218.

V. GRAPHS AND GRAPHICAL SOLUTIONS OF QUADRATICS[*]

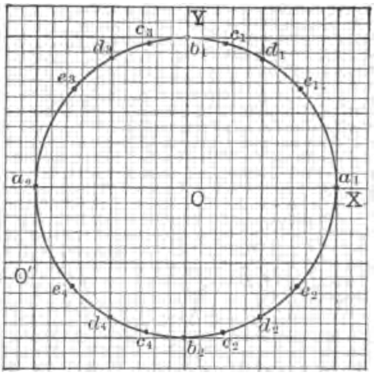
221. The Various Graphs of Equations of the Second Degree.

1. Construct the graph of the equation $x^2 + y^2 = 4$.

Solution

$$x^2 + y^2 = 4$$

(x, y)	Points
$(\pm 2, 0)$	a_1, a_2
$(0, \pm 2)$	b_1, b_2
$(\pm .5, \pm 1.9)$	c_1, c_2, c_3, c_4
$(\pm 1, \pm 1.7)$	d_1, d_2, d_3, d_4
$(\pm 1.5, \pm 1.3)$	e_1, e_2, e_3, e_4



a. The student may not see how $(0, \pm 2)$ gives *two* points, or how $(\pm 1, \pm 1.7)$ gives *four* points. He should observe that (0, ± 2) is really the two points, (0, 2) and (0, − 2); also that (±1, ± 1.7) denotes four points, viz. (1, 1.7), (1, −1.7), (− 1, 1.7), (− 1, − 1.7). Notice that the coördinates of each of these six points satisfy the equation $x^2 + y^2 = 4$.

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[*] If desired the sections in Chapter XVI on *the Discussion of the Quadratic Equation* and on *Reasoning in Equations* can be taken up here.

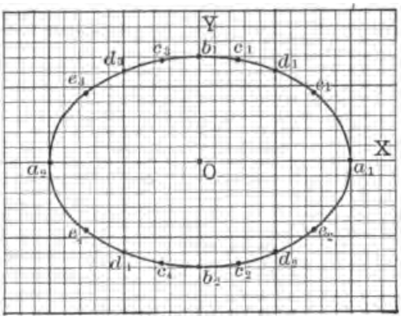
Construct now the graph of (1) $x^2 + y^2 = 9$.

From these examples we readily see that the graph of an equation of the form $x^2 + y^2 = r^2$ *is a* **circle** *whose radius is* r.

Hence, graph of an equation of the form $x^2 + y^2 = r^2$ can be immediately constructed by using r as radius and origin as center.

Construct the graphs for the following equations:

(2) $x^2 + y^2 = 16$. (3) $x^2 + y^2 = 12$. (4) $x^2 + y^2 = 1$.



2. Construct the graph of the equation $x^2 + 2y^2 = 4$.

SOLUTION

$x^2 + 2y^2 = 4$

(x, y)	POINTS
$(\pm 2, 0)$	a_1, a_2
$(0, \pm 1.4)$	b_1, b_2
$(\pm .5, \pm 1.4^-)$	c_1, c_2, c_3, c_4
$(\pm 1, \pm 1.2)$	d_1, d_2, d_3, d_4
$(\pm 1.5, \pm .9^+)$	e_1, e_2, e_3, e_4

b. **Meaning of Imaginary Values for *x* and *y* in finding Coördinates for Graphs.** Suppose we assign the value $x = 3$ in the equation just used, $x^2 + 2y^2 = 4$. Making this substitution, we get $y = \sqrt{-2.5}$. Thus, for $x = 3$, y is **imaginary**. This means that there are no points of the curve as far from the origin as $x = +3$ or $x = -3$. The curve stops, in fact, with the value $x = +2$ at the right, and with $x = -2$ at the left. Similarly, if we put $y = \pm 2$, we get $x = \sqrt{-4}$. We see from the diagram that there are no points of the curve above $y = +1.4$, and none below $y = -1.4$.

Construct the graphs of the following equations:

(1) $x^2 + 4y^2 = 5$. (2) $3x^2 + 4y^2 = 8$. (3) $10x^2 + y^2 = 10$.

c. These graphs, or curves, are called **ellipses**. They occur as the graphs of all equations of the form $ax^2 + by^2 = c$, that is, of all equations in which the coefficients of x^2 and y^2 are unequal, and the sign between the terms in the left member is +.

If the coefficients of x^2 and y^2 are close together in value, to what curve does the ellipse approximate? What happens if they are quite unequal?

The earth moves round the sun in an **elliptical** orbit, nearly circular. Halley's comet, on the other hand, moves in a very long ellipse.

(4) $x^2+9y^2=18$. (5) $10x^2+12y^2=24$. (6) $9x^2+12y^2=1$.

3. Construct the graph of the equation $x^2-2y^2=4$.

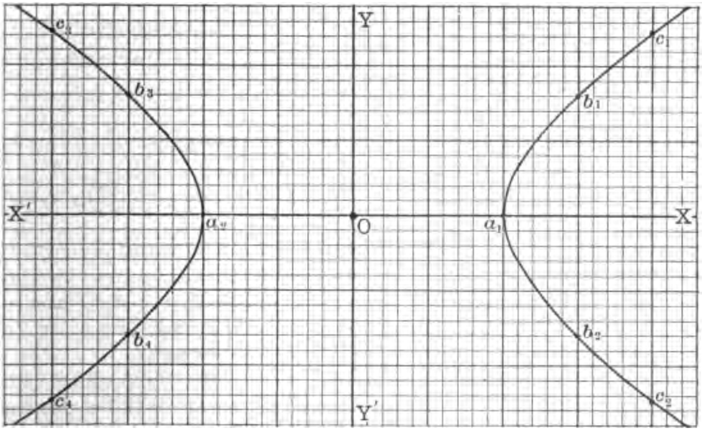
SOLUTION

$$x^2-2y^2=4$$

(x, y)	POINTS
$(\pm 2, 0)$	a_1, a_2
$(0, \text{imag.})$	
$(\pm 1, \text{imag.})$	
$(\pm 3, \pm 1.6)$	b_1, b_2, b_3, b_4
$(\pm 4, \pm 2.5^-)$	c_1, c_2, c_3, c_4
$(\pm 5, \pm 3.2^+)$	d_1, d_2, d_3, d_4

d. This curve is called an **hyperbola.** An examination of the values in the table, noting from the equation how they will continue to increase indefinitely, shows that one branch of the curve will continue out indefinitely to the northeast (to follow the customary map rules for directions), another to the southeast, another to the northwest, and a fourth to the southwest.

The *imaginary* values of y show that no points of the curve lie inside of the region inclosed by the lines $x=+2$, and $x=-2$.



Notice that the equation used is identically the same as that of Ex. 2 except that + in the ellipse equation is replaced by − in that of the hyperbola.

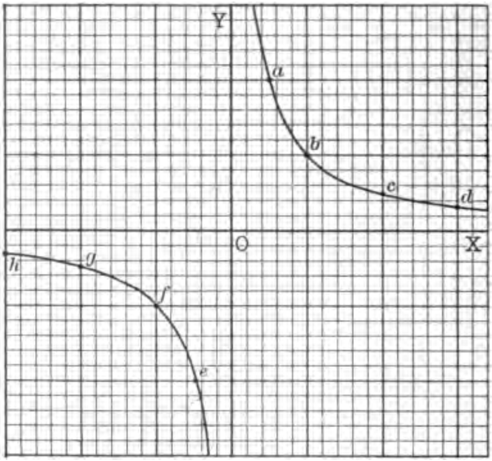
The hyperbola is the orbit of comets which come into the solar system, go round the sun, and then go out again never to return.

Construct the graphs of the following equations:

(1) $x^2 - 4y^2 = 4$. (2) $4x^2 - y^2 = 16$. (3) $2y^2 - 3x^2 = 6$.

(4) $x^2 - y^2 = 9$. (5) $y^2 - x^2 = 9$. (6) $x^2 - 10y^2 = 1$.

4. Construct the graph of the equation $xy = 1$.



Solution

$xy = 1$

(x, y)	Pts.
$(.5, 2)$	*a*
$(1, 1)$	*b*
$(2, .5)$	*c*
$(3, .3+)$	*d*
. . . . . .	. .
$(-.5, -2)$	*e*
$(-1, -1)$	*f*
$(-2, -.5)$	*g*
$(-3, -.3+)$	*h*
. . . . . .	. .

Construct the graphs of the following equations:

(1) $xy = 10$. (2) $xy = 1$. (3) $xy = -2$. (4) $xy = 50$.

e. These graphs are hyperbolas, though coming from equations quite different from those in 3. They differ from the hyperbolas we have just been constructing in having their "axes" extend diagonally instead of east and west, or north and south. This is the graph for inverse proportion (see § 243). Evidently x increases as y diminishes, and *vice versa.*

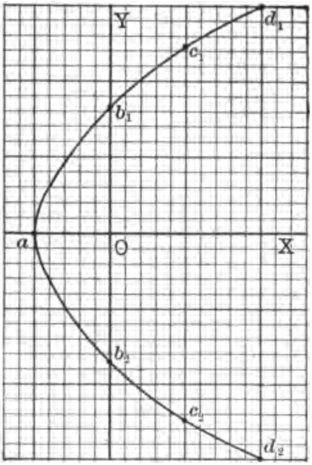
The graphs just constructed can be used to read off corresponding values for quantities which are *inversely* proportional just as we read off corresponding values for quantities which were *directly* proportional in § 122. Thus, if a man is paid $10 for doing a job of work, x is the number of days he works, and y is the wages per day, then the graph for $xy = 10$ (see eq. (1) above) will give a value for x corresponding to any assigned value of y, or a value for y corresponding to any assigned value of x.

5. Construct the graph of the equation $y^2 = 3x + 3$.

SOLUTION

$$y^2 = 3x + 3$$

(x, y)	PTS.
$(-1, 0)$	a
$(0, \pm 1.7)$	b_1, b_2
$(1, \pm 2.5^-)$	c_1, c_2
$(2, \pm 3)$	d_1, d_2
$(6, \pm 4.6^-)$	e_1, e_2
. . . .	. . .



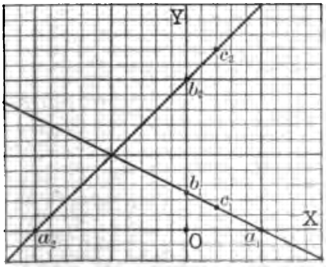
f. **This graph is called a parabola. It has apparently something like the same shape as the hyperbola, but it has only *one* branch. If x is put equal to -1.1 or any numerically larger negative number, y is imaginary. The parabola is the path of projectiles like bullets and is the curve of headlight reflectors.**

Construct the graphs of the following equations:

(1) $y^2 = 2x + 4$. (2} $y^2 = 2x - 4$. (3) $y^2 = x$.
(4) $x^2 = 2y + 4$. (5) $x^2 = 12y - 3$ (6) $x^2 = y$.

6. Construct the graph of the equation $(x - y + 2)(x + 2y - 1) = 0$, or $x^2 + xy - 2y^2 + x + 5y - 2 = 0$.

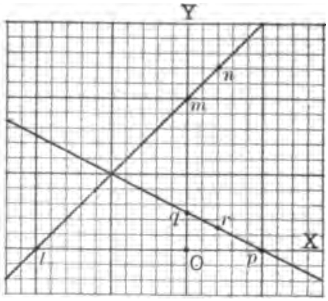
Presumably this is the equation of *the two straight lines*, $x - y + 2 = 0$, and $x + 2y - 1 = 0$, since values that satisfy either of them satisfy the given equation. (See § 116, 5.)



SOLUTION

$$x^2 + xy - 2y^2 + x + 5y - 2 = 0$$

(x, y)	PTS.
$(1 \text{ or } -2, 0)$	a_1, a_2
$(0, .5 \text{ or } 2)$	b_1, b_2
$(.4, .3 \text{ or } 2.4)$	c_1, c_2



Solution

$x - y + 2 = 0$

$(-2, 0)$	l
$(0, 2)$	m
$(.4, 2.4)$	n

$x + 2y - 1 = 0$

$(1, 0)$	p
$(0, .5)$	q
$(.4, .3)$	r

g. We see from the diagrams that a_1, a_2, b_1, b_2, c_1, c_2, are all on line ln or line pq. This rule would hold if any number of points were obtained.

The pupil may have some trouble in seeing how the coördinates of a_1, a_2, b_1, b_2, c_1, c_2, were found. Notice if $y=0$, the equation reduces to $x^2+x-2=0$, whose roots by factoring are 1 and -2. Again, if $x = 0$, the equation reduces to $2y^2 - 5y + 2 = 0$, whose roots are 2 and .5. If $x = .4$, the equation reduces to $y^2 - 2.7y + .72 = 0$, whose roots are 2.4 and .3. If $x = 2$, the equation reduces to $2y^2 - 7y - 4 = 0$, whose roots are 4 and $-.5$. As a rule in such cases, the factoring method cannot be used, and we resort to that of § 210.

222. Effect of Change of Origin on Equations. Reverting to the figure accompanying Ex. 1, § 221, we see that if O' is taken as the origin instead of O, every abscissa will then be increased by 2 over what it was, and every ordinate by 1. The equation of the circle now becomes

$$(x-2)^2 + (y-1)^2 = 4,$$

or

$$x^2 + y^2 - 4x - 2y + 1 = 0.$$

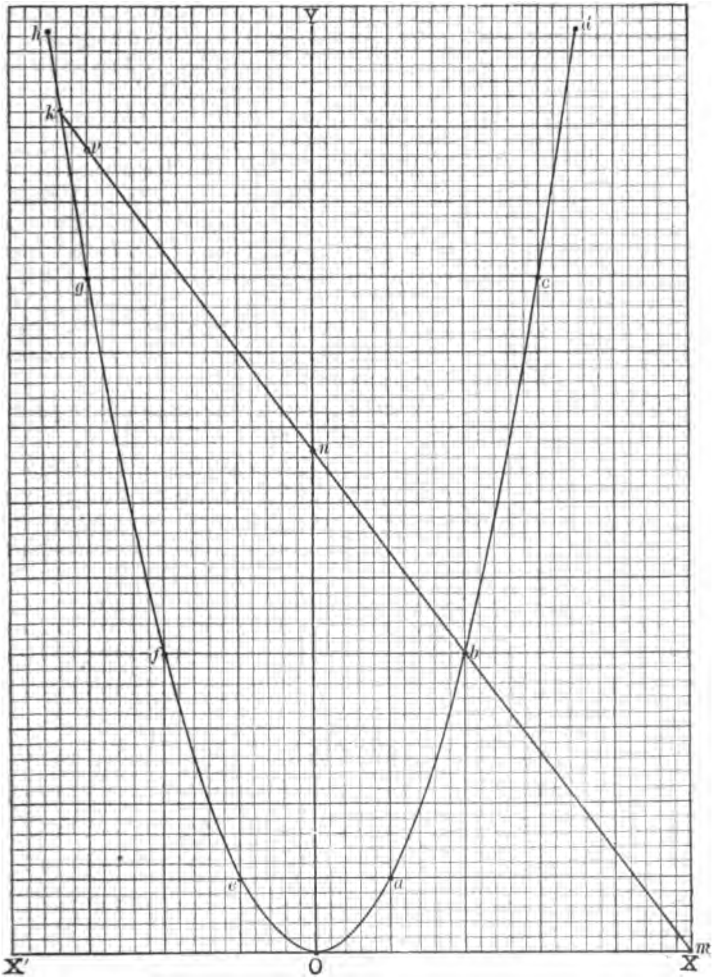
Thus we see that an equation of this form, whose coefficients of x^2 and y^2 are equal, gives a circular graph, but one in which origin is not at its center.

A similar change could be made in the equations of the other graphs without changing the graphs themselves.

223. Graphical Solution of a Quadratic Equation containing One Unknown Quantity. To construct a graph two unknowns are necessary. In the present problem one of these has to be *introduced.*

1. Given $3x^2 + 4x = 20$, to find values of x by graph method.

Solution. Let (1) $y = x^2$; then substituting $y = x^2$ in the given equation, we have (2) $3y + 4x = 20$. (Solution continued on page 188.)



a. **A piece of fine black thread can be laid so as to pass through the several points located on the squared paper. The position the thread takes when it passes through the points locates the graph very closely.**

(1) $y = x^2$.

(x, y)	Points
$(0, 0)$	o
$(\pm 1, 1)$	a, e
$(\pm 2, 4)$	b, f
$(\pm 3, 9)$	c, g
$(\pm 3.5, 12.25)$	d, h
(etc., etc.)	etc.

(2) $3y + 4x = 20$.

(x, y)	Pts.
$(0, 6.7)$	n
$(5, 0)$	m
$(-3, 10.7)$	p

We must now construct the graphs for equations (1) and (2). Values are assigned to y and the corresponding values of x are calculated as in Chapter IV.

Locating the several points for equation (1) on the squared paper with reference to two axes, marking them with their letters, and joining them by a running line, we have the figure as shown in the diagram, p. 187. Joining the points located by equation (2) there results the straight line *mnp*. These two graphs *cross* at b and k, or at $x = 2$, and $x = -3.4^-$, which are the roots sought.

Check. Solving $3x^2 + 4x = 20$ by completing the square as in § 210, we have $x = 2$, and $x = -3.33^+$.

b. The graph of $y = x^2$ is (§ 221, *f*) a **parabola**. The unfinished ends are known to extend on indefinitely. Have students practice making this graph.

c. Observe that the curve $y = x^2$, once constructed, can be used in the solution of any quadratic containing one unknown. It only remains in each particular problem to locate the straight line by means of its Eq. (2). Where this straight line crosses the parabola are found the values of x sought.

Construct the parabola with ink and the straight line with lead pencil. Then the lead pencil mark can be erased when one begins a new problem.

Solve the following equations by the graph method, and check by completing the square method:

2. $x^2 - 2x - 3 = 0$.
3. $3x^2 - 2 = 5x$.
4. $12x^2 + x = 6$.
5. $8x^2 - 17x = 115$.
6. $7x^2 - 39 = 8x$.
7. $10 + 16x = -6x^2$.
8. $3x^2 - 2x = 7$.
9. $7x^2 + 14x = 21$.
10. $5x^2 - 20 = 0$.
11. $7x^2 + 5x = 31$.
12. $3x^2 + 4x = 4$.
13. $5x^2 - 7x = 1$.

14. Solve by the diagram $x^2 = 7$, *i.e.* extract square root of 7.

Solution. Corresponding to $y = 7$ is $x = 2.6^+$. *Ans.*

15. Find by the diagram $\sqrt{5}$; $\sqrt{2.8}$; $\sqrt{7}$; $\sqrt{12}$; $\sqrt{.9}$; $\sqrt{6.4}$.

16. The diagram can also be used to square numbers. Thus corresponding to $x = 1.4$ is $y = 2^-$, which is the square of 1.4.

17. Find by the diagram 1.8^2; 2.5^2; 75^2 (or $.75^2$); 3.4^2.

224. Solution of Simultaneous Equations by Means of Graphs.

1. Solve $\left\{\begin{array}{l}(1)\ x^2+y^2=9\\(2)\ x+y=2\end{array}\right\}$ by the graph method.

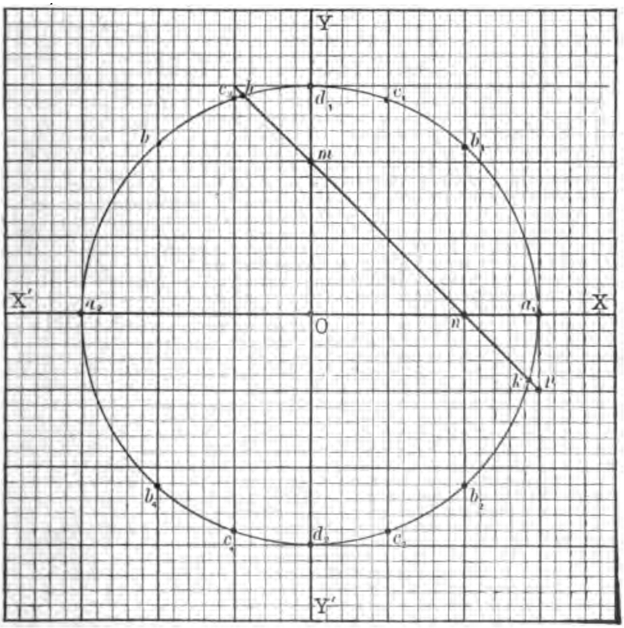
(1) $x^2+y^2=9$

$(\pm 3, 0)$	a_1, a_2
$(\pm 2, \pm 2.2)$	b_1, b_2, b_3, b_4
$(\pm 1, \pm 2.8)$	c_1, c_2, c_3, c_4
$(0, \pm 3)$	d_1, d_2

(2) $x+y=2$

$(0, 2)$	m
$(2, 0)$	n
$(3, -1)$	p

Locating the several points and joining them, we have the graph as represented in the figure. Evidently the graph of $x^2+y^2=9$ is the circumference of a circle whose radius is $\sqrt{9}$, or 3.



The graphs *cross* at k and h, whose coördinates are

$$\left\{\begin{array}{l}x=2.9,\\y=-.9,\end{array}\right. \text{ and } \left\{\begin{array}{l}x=-.9,\\y=2.9.\end{array}\right.$$

To check these answers, the problem is solved by the method of § 219, whence is found $\begin{cases} x = 2.87+, \\ y = -.87+, \end{cases}$ and $\begin{cases} x = -.87+, \\ y = 2.87+. \end{cases}$

Solve the following by the graph method, and check by a purely algebraic solution. The graphs of equations like $x^2 + y^2 = 9$ can be constructed very quickly with a compass. How is the radius found in each case?

2. $\begin{cases} x^2 + y^2 = 36, \\ x + y = 8. \end{cases}$ 3. $\begin{cases} x^2 + y^2 - 40 = 0, \\ 3x - y = 6. \end{cases}$ 4. $\begin{cases} x^2 + y^2 = 37, \\ x - y = 5. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 25, \\ 4x + 3y = 20. \end{cases}$ 6. $\begin{cases} x^2 + y^2 = 169, \\ 3x + 9 = 2y. \end{cases}$ 7. $\begin{cases} x^2 + y^2 = 121, \\ 5x + y = 46. \end{cases}$

8. $\begin{cases} x^2 + 2y^2 = 34, \\ x + y = 7. \end{cases}$

SUGGESTION. The graph of $x^2 + 2y^2 = 34$, as explained in § 221, 2, is an *oval*, or an *ellipse* in shape. The method of solution is precisely the same as in Ex. 1.

9. $\begin{cases} 4x^2 + y^2 = 25, \\ 2x + y = 7. \end{cases}$ 10. $\begin{cases} 9x^2 + 4y^2 = 72, \\ 5x - 3y = 1. \end{cases}$

11. $\begin{cases} 3x^2 + 4y^2 = 43, \\ 2x - y = 4. \end{cases}$ 12. $\begin{cases} 2x^2 + y^2 = 33, \\ x - 3y = 1. \end{cases}$

13. $\begin{cases} 2x^2 - xy + y^2 = 7, \\ 4x - 3y = 5. \end{cases}$ 14. $\begin{cases} 3x^2 + 2xy + 2y^2 = 35, \\ 4x - 5y = 7. \end{cases}$

15. $\begin{cases} (1)\ x^2 - 2y^2 = 1, \\ (2)\ x + 6y = 3. \end{cases}$

SOLUTION. The graph for equation (1) is (§ 221, *d*) different from the preceding ones, owing to the *minus* sign before $2y^2$. The method of solution, however, is precisely the same as before. Note that $(\pm 2, \pm 1.2)$ gives *four* points.

(1) $x^2 - 2y^2 = 1$

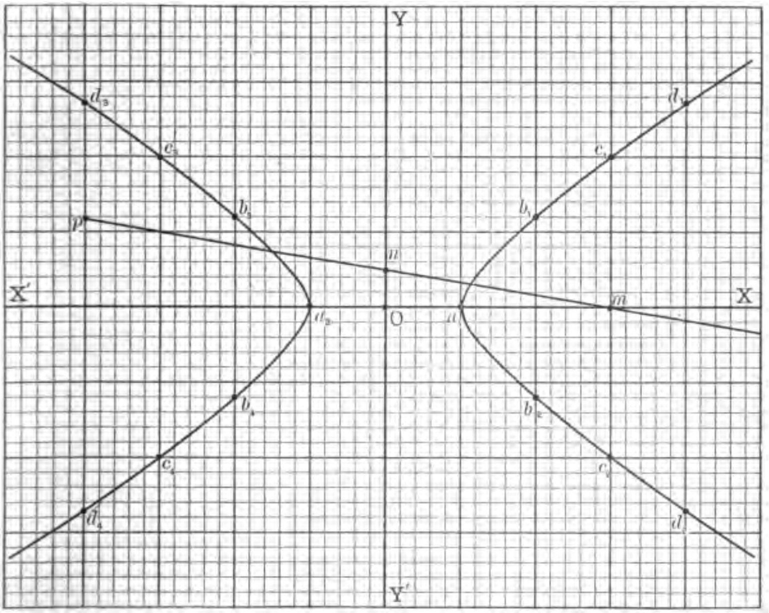
$(\pm 1, 0)$	a_1, a_2
$(\pm 2, \pm 1.2)$	b_1, b_2, b_3, b_4
$(\pm 3, \pm 2)$	c_1, c_2, c_3, c_4
$(\pm 4, \pm 2.7)$	d_1, d_2, d_3, d_4
(etc., etc.,)	etc.

(2) $x + 6y = 3$

$(3, 0)$	m
$(0, .5)$	n
$(-4, 1.2)$	p

Constructing the graphs through the located points, we have the accompanying diagram. The four unfinished ends of the curved graph extend out indefinitely. The curve, it is clear, consists of two parts. It is an **hyperbola**.

The straight line graph of equation (2) crosses the right-hand branch of the hyperbola at $x = 1.1$, $y = .3^+$, and the left-hand branch at $x = -1.5^-$, $y = .7^+$. Checking the answer by a purely algebraic solution (§ 218), we get the same results.



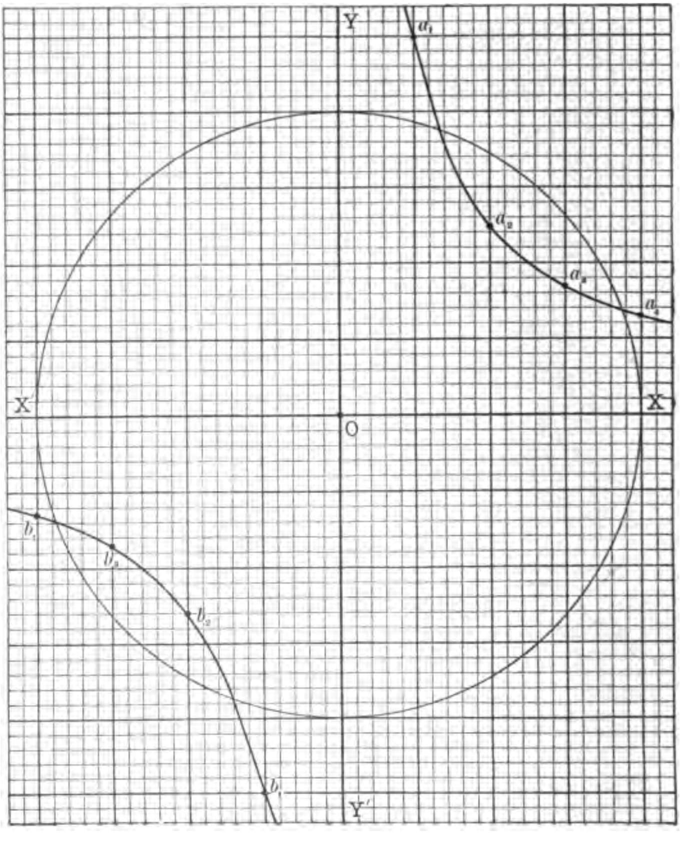
16. $\begin{cases} (1)\ x^2 + y^2 = 16, \\ (2)\ xy = 5. \end{cases}$

Solution. The graph of (1) is, of course, a circle whose radius is 4. The graph of

(2) $xy = 5$		$(-1,\ -5)$	b_1
$(1,\ 5)$	a_1	$(-2,\ -2.5)$	b_2
$(2,\ 2.5)$	a_2	$(-3,\ -1.7)$	b_3
$(3,\ 1.7)$	a_3	$(-4,\ -1.25)$	b_4
$(4,\ 1.25)$	a_4	etc., etc.	etc.

equation (2) **is an hyperbola**, but turned through an angle of 45°, as compared with that of Ex. 15.

Solve by algebra as well as by graph, being careful to get four values for x and four for y.



17. $\begin{cases} 3x^2 - y^2 = 47, \\ x + y = 9. \end{cases}$

18. $\begin{cases} x + 2y = 9, \\ 3y^2 - 5x^2 = 43. \end{cases}$

19. $\begin{cases} 2x^2 - 3xy + y^2 = 8, \\ x + y = 4. \end{cases}$

20. $\begin{cases} x^2 - 2y^2 = -7, \\ x + y = 9. \end{cases}$

21. $\begin{cases} 2y^2 - x^2 = -23, \\ x - 2y = 3. \end{cases}$

22. $\begin{cases} 3x^2 - 4y^2 = 11, \\ 4x - y = 10. \end{cases}$

23. $\begin{cases} x^2 + y^2 = 9, \\ xy = 2. \end{cases}$

24. $\begin{cases} 3x^2 + 2y^2 = 6, \\ 3xy = 1. \end{cases}$

25. $\begin{cases} 4x^2 + 9y^2 = 73, \\ xy = 4. \end{cases}$

26. $\begin{cases} (1)\ x^2 - y = 7, \\ (2)\ 3x - y = 9. \end{cases}$

Suggestion. The graph of (1) has the shape of that of § 223, but is pushed down so that the curve crosses the axis of X.

27. $\begin{cases} x^2 - 3y = 19, \\ x - y = 3. \end{cases}$

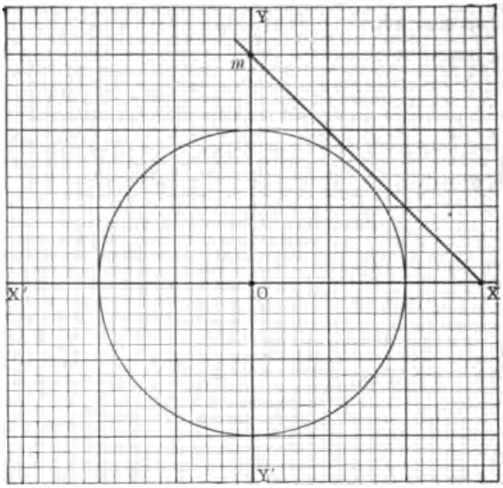
28. $\begin{cases} x - 2y^2 = 1, \\ x - 3y = 6. \end{cases}$

29. $\begin{cases} x^2 - 2y = 7, \\ xy = 3. \end{cases}$

a. An equation of the form $ax^2 + ay^2 = b$ has a circle for its graph.
An equation of the form $ax^2 + by^2 = c$ has an ellipse for its graph.
An equation of the form $ax^2 - by^2 = c$ has an hyperbola for its graph.
An equation of the form $xy = a$ has an hyperbola for its graph.
An equation of the form $ax^2 + by = c$ has a parabola for its graph.
An equation of the form $ax + by^2 = c$ has a parabola for its graph.

225. Graphical Significance of Imaginaries (§ 199) in the Solution of Equations. Whenever upon solving two simultaneous equations the roots come out imaginary, it will be found that the graphs of the two equations *do not intersect at all.* See § 221, *b.*

1. $\begin{cases} x^2 + y^2 = 4, \\ x + y = 3. \end{cases}$



Constructing the graphs, we have the figure as represented, the graphs not intersecting. Solving by a purely algebraic method, we get,

$$x = \frac{3 \pm \sqrt{-1}}{2}, \quad y = \frac{3 \mp \sqrt{-1}}{2}.$$

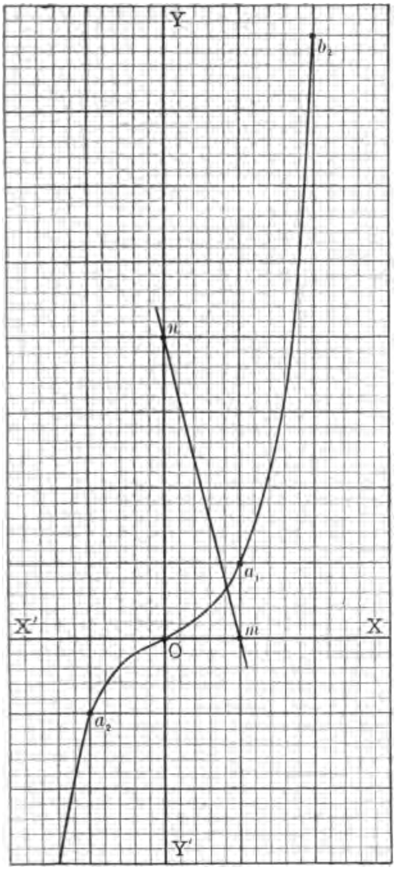
Construct the graphs in the following, solving also by algebra and verifying:

2. $\begin{cases} x^2 + y^2 = 36, \\ x - 2y = 14. \end{cases}$ 3. $\begin{cases} 16x^2 - 9y^2 = 144, \\ 6x = y. \end{cases}$ 4. $\begin{cases} xy = 5, \\ x + y = 4. \end{cases}$

226. Cubic Equations, or Equations of the Third Degree, containing One Unknown, solved by the Graph Method of § 223.

1. $x^3 + 4x - 4 = 0.$

Let, (1) $y = x^3$; then, on substituting, (2) $y + 4x = 4$.



(1) $y = x^3$.

(0, 0)	0
(1, 1)	a_1
(− 1, − 1)	a_2
(2, 8)	b_1
(− 2, − 8)	b_2
(3, 27)	c_1
(− 3, − 27)	c_2

(2) $y + 4x = 4$.

(1, 0)	m
(0, 4)	n

Constructing the two graphs from the tabulated data, we see that they intersect at $x = .8+$.

2. $3x^3+4x-7=0$.

3. $5x^3+12x=15$.

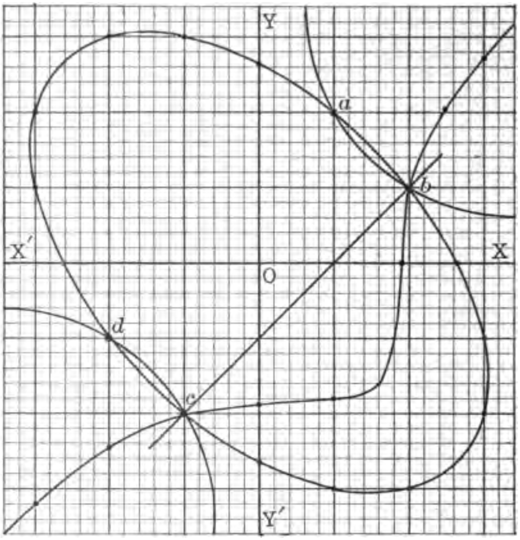
4. $4x^3-19x+42=0$.

5. The graph of this article can be used to extract the cube root and to cube numbers. See § 223, 14–17.

227. Equations of any Degree in One or Two Unknowns can be solved by the graph method. However, the complexity of the shape of the graph rapidly increases when the degree of the equation goes above the second degree, with resulting difficulty in the construction.

Light is thrown on the solution of systems of equations by the use of graphs. We shall now consider an example of such graphical explanation, showing how the various graphs in a system intersect one another.

228. Comparison of the Algebraic and Graphical Solutions of the System (1) $x^3-y^3=7$, (2) $x-y=1$.



$$\begin{array}{lrl}
(3) & x^2+\quad xy+y^2=7 & [(1)\div(2)\text{ Div. Ax.}] \\
(1_1) & \underline{x^2-2\,xy+y^2=1} & (\text{Ax. ?}) \\
(4) & 3\,xy\quad\;\;=6 & (\text{Ax. ?}) \\
(4_1) & xy\quad\;\;=2 & (\text{Ax. ?}) \\
(3) & \underline{x^2+\quad xy+y^2=7} & \\
(5) & x^2+2\,xy+y^2=9 & (\text{Ax. ?}) \\
(5_1) & x+y=\pm 3 & (\text{Ax. ?}) \\
(2) & \underline{x-y=\quad 1} & \\
 & x=\quad 2,\text{ or }-1 & (\text{Ax. ?}) \\
 & y=\quad 1,\text{ or }-2 & (\text{Ax. ?})
\end{array}$$

In the diagram above notice that the graph of $x^3-y^3=7$ is the curve resembling that of § 226. The equation $x^2+xy+y^2=7$ gives the ellipse. The equation $xy=2$ gives the hyperbola. The equation $x - y = 1$ gives, of course, the straight line. The equation $(x - y)^2 = 1$ represents two straight lines (§ 221, 6). All these curves intersect in $(x = 2, y = 1)$, $(x = -1, y = -2)$. The ellipse and hyperbola intersect also in $(x = 1, y = 2)$, $(x = -2, y = -1)$. The first pair of values of x and y satisfy *all* the equations.

If one pair of equations have the same points of intersection as another pair, the two systems of equations (§ 148) are said to be **equivalent**. The diagram shows quickly to the eye which systems of the preceding sets of equations are equivalent.

VI. HARDER QUADRATICS*

229. Solutions of Harder Quadratics. The variety of forms taken by quadratic equations, or equations quadratic in character, especially simultaneous quadratics, is large. To become expert in the solution of such equations requires considerable experience, and this necessitates time to acquire it. The practical value of such skill is not very great, since there is little call in the applications of mathematics for the solution of such equations. If the student desires to prepare himself as quickly as possible for solving such problems, his best plan is to try to get a grasp of the various *types* of problems, *methods* of solutions, and *special devices* that are commonly met with in textbooks. For these reasons, in what follows solutions and suggestions are offered freely, but the number of exercises given is small.

230. Solutions of Quadratics containing one Unknown Quantity. Classes of Problems.

I. *Equations quadratic in character containing fractional exponents.*

1. Solve $x^{\frac{1}{2}} + 4x^{\frac{1}{4}} = 21$, and verify.

SOLUTION. — Notice that $x^{\frac{1}{2}} = (x^{\frac{1}{4}})^2$. Thus, this equation contains a certain unknown in one term, and the *square* of this unknown in another term. If we put $x^{\frac{1}{4}} \equiv z$, then $x^{\frac{1}{2}} \equiv z^2$. Making these substitutions, the given equation becomes $z^2 + 4z = 21$. Thus, we see that this equation can easily be solved by any of the methods of solving a quadratic.

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* **This section can be abridged as much as desired. Little that follows depends on it.**

Solving the equation by factoring (§ 208), we get

$$(x^{\frac{1}{4}} + 7)(x^{\frac{1}{4}} - 3) = 0.$$

$$\therefore x^{\frac{1}{4}} = -7, \text{ and } x^{\frac{1}{4}} = 3. \qquad (\text{Ax.?})$$

$$\therefore x = 2401, \text{ and } x = 81. \qquad (\text{Ax.?})$$

VERIFICATION. $(2401)^{\frac{1}{2}} + 4(2401)^{\frac{1}{4}} \neq 21$; $(81)^{\frac{1}{2}} + 4(81)^{\frac{1}{4}} \equiv 21$.

Thus, $x = 2401$ is *not* a root of this equation if positive square roots are taken, while 81 is a root.

2. $x + 4x^{\frac{1}{2}} = 5$.
3. $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} = -6$.
4. $x^3 - 24x^{\frac{3}{2}} = 81$.
5. $7\sqrt[3]{z^2} + \sqrt[3]{z} = 350$.
6. $12x^{-\frac{3}{4}} - x^{-\frac{3}{8}} = \frac{1}{16}$.
7. $ax^{2n} + bx^n = c$.
8. $x^{-\frac{1}{2}} - 6x^{\frac{1}{2}} = 1$.
9. $x^{\frac{3}{2}} - 4x = 5x^{\frac{1}{2}}$. SUGGESTION. Divide through by $x^{\frac{1}{2}}$.

II. *Equations solved first for functions of the unknown.*

1. Solve $x^2 = 9 + \sqrt{x^2 - 3}$, and verify.

SOLUTION. In such problems the radical quantity is taken as the unknown at first. By adding to both members just the right quantity the equation can be put in the quadratic form. Thus, we can write here

$$x^2 - 3 - \sqrt{x^2 - 3} = 6. \quad (\text{Sub. Ax.})$$

To see more clearly that this is quadratic in character, put $\sqrt{x^2 - 3} \equiv z$. Then $x^2 - 3 = z^2$, and the equation becomes $z^2 - z = 6$.

Then $(z - 3)(z + 2) = 0$. (Factoring.)

Whence $z = 3$, and $z = -2$.

Now, we have $\sqrt{x^2 - 3} = 3$, or -2,

whence $x^2 = 12$, or 7,

and $x = 2\sqrt{3}$, or $\sqrt{7}$. *Ans.*

VERIFICATION. $(2\sqrt{3})^2 \equiv 9 + \sqrt{(2\sqrt{3})^2 - 3}$; $(\sqrt{7})^2 \neq 9 + \sqrt{(\sqrt{7})^2 - 3}$.

Thus, we see the root $\sqrt{7}$ does *not* verify in the original equation with the sign of the radical taken positive.

2. $x^2 - 6 = 2\sqrt{x^2 + 9}$.
3. $x^2 = \sqrt{x^2 - 7} + 13$.
4. $x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24$.
5. $x^2 - x - \sqrt{x^2 - x + 4} = 8$.
6. $2\sqrt{x^2 - 3x + 11} = x^2 - 3x + 8$.
7. $\sqrt{x^2 - 8x + 31} + (x - 4)^2 = 5$.

8. $x^4+x^3-4x^2+x+1=0$.

Solution. $x^2+x-4+\frac{1}{x}+\frac{1}{x^2}=0$, (Div. Ax.)

or, $\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-4=0$, (Arranging.)

Then, $\left(x^2+2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-6=0$, (Adding 2 in one term and subtracting it in another to make first quantity a perfect square).

Now put $z\equiv x+\frac{1}{x}$. Then, we have

$$z^2+z-6=0,$$

Whence $z=+2$ or -3.

We now have $x+\frac{1}{x}=2$, and $x+\frac{1}{x}=-3$.

Solving these equations gives the four values of x sought.

9. $16y^2+\frac{1}{y^2}=28$.

10. $\left(x+\frac{8}{x}\right)^2+x=42-\frac{8}{x}$.

11. $\frac{1}{(2x-4)^2}=\frac{1}{8}+\frac{2}{(2x-4)^4}$.

Suggestion. Put $z\equiv(2x-4)^2$.

12. $\frac{x^2+x}{2}+\frac{2}{x^2+x}=2$.

13. $\frac{x^2+1}{x-1}-\frac{4(x-1)}{x^2+1}=\frac{21}{5}$.

Suggestion to Ex. 13. Put $z\equiv\frac{x^2+1}{x-1}$.

14. $x^4-2x^3-2x^2+3x-108=0$.

Solution. To find a function that can be used we extract the square root of the left member (§ 171). This gives x^2-x for the root, and $-3x^2+3x-108$ for the remainder. Hence the given equation can be written

$$(x^2-x)^2-3(x^2-x)-108=0,$$

which equation can now be solved for x^2-x first, and then for x.

15. $x^4-6x^3+5x^2+12x-60=0$.

16. $4x^4+\frac{x}{2}=4x^3+1$.

III. *Tartaglia's solution of the cubic by means of a quadratic equation.*

1. Given $x^3-6x-9=0$, to find x.

Solution. $(y+z)^3-6(y+z)-9=0$. ($y+z$ put for x.)

$y^3+z^3+3yz(y+z)-6(y+z)-9=0$. ($y+z$ cubed and arranged.)

$(y^3+z^3)+(3yz-6)(y+z)-9=0$. (Factoring middle terms, § 77.)

Since there are now *two* unknowns, and there was only one, x, at first, we may choose a relation between y and z expressed in an equation. Now, we see if we put $3yz - 6 = 0$, we greatly simplify the last equation, all of the terms except $y^3 + z^3 - 9$ dropping out. Then we have

$$(1)\ 3yz - 6 = 0,\ \text{or}\ z = \frac{2}{y},\ \text{and}\ (2)\ y^3 + z^3 - 9 = 0.$$

Substituting $z = \frac{2}{y}$ in (2), this equation gives (using factoring method), $y^3 = 8$, or 1, whence $y = 2$, or 1.

Then, $z = 1$, or 2, and $x = y + z = 3$. *Ans.*

The equation containing the other two roots can be found by dividing $x^3 - 6x - 9$ by $x - 3$ and setting the quotient equal to 0. (§ 215, 25).

2. $x^3 - 18x - 35 = 0$. 3. $x^3 - 10x + 24 = 0$.

a. A curious fact about Tartaglia's solution is that it gives a usable result only when one of the roots of the given equation is real and the other two are *imaginary*. When all the roots are real, resort may be had to the divisibility theorem (§ 69) to find the first of the three roots.

231. Solution of Harder Simultaneous Quadratics. Besides the two most common cases, or kinds of problems, in simultaneous quadratics described in §§ 218, 219, a number of others are also usually given. The number of *devices* used in solving these equations one time and another is very large.

I. *When each equation is of the form* $ax^2 + by^2 = c$.

1. (1) $9x^2 + 25y^2 = 225$; (2) $x^2 + y^2 = 9$.

2. (1) $4x^2 - 9z^2 = 36$; (2) $x^2 + z^2 = 15\frac{1}{2}$.

II. *When all the terms containing unknowns in both equations are of the second degree,* often called the **homogeneous case.**

$$1.\ \begin{cases} (1)\ 4x^2 - xy - y^2 = -16, \\ (2)\ 3xy + y^2 = 28. \end{cases}$$

Solution. To solve these equations we follow the very unusual course of eliminating the *known* terms. The process may seem a little more natural if we insert z^2 as the literal part in the right members and then eliminate z^2. Notice this would make *all* the terms in the equations homogeneous.

$$\begin{array}{lrl}
(1_1) & 28x^2 - 7xy - 7y^2 = -112z^2 & \text{(Eq. (1)} \times 7.) \\
(2_1) & 12xy + 4y^2 = 112z^2 & \text{(Eq. (2)} \times 4.) \\
\hline
(3) & 28x^2 + 5xy - 3y^2 = 0 & \text{(Ax. ?)}
\end{array}$$

$$(4x - y)(7x + 3y) = 0. \qquad \text{(Factoring.)}$$

Then, $4x - y = 0,$ whence $y = 4x;$

and, $7x + 3y = 0,$ whence $y = -\frac{7}{3}x.$

Now, (2) $3x(4x) + (4x)^2 = 28.$ (Putting $4x$ for y.)

$$\left.\begin{array}{l} \therefore x = \pm 1, \\ y = 4 \times \pm 1 = \pm 4 \end{array}\right\} \textit{Ans.} \quad \begin{array}{l} \text{(Since } 28x^2 = 28.) \\ \text{(Since } y = 4x.) \end{array}$$

and

Also, (2) $3x(-\frac{7}{3}x) + (-\frac{7}{3}x)^2 = 28.$ (Putting $y = -\frac{7}{3}x$.)

$$\left.\begin{array}{l} \therefore x = \pm\sqrt{-18} = \pm 3\sqrt{-2}, \\ y = -\frac{7}{3} \times \pm 3\sqrt{-2} = \mp 7\sqrt{-2} \end{array}\right\} \textit{Ans.} \quad \begin{array}{l} \text{(Since } -\frac{14}{9}x^2 = 28.) \\ \text{(Since } y = -\frac{7}{3}x.) \end{array}$$

and

a. In the verification notice the upper signs go together and the lower signs together. Thus, $x = +3\sqrt{-2}$ goes with $y = -7\sqrt{-2}$.

2. $\begin{cases} 3x^2 - 4xy + 2y^2 = 17, \\ y^2 - x^2 = 16. \end{cases}$ 3. $\begin{cases} x^2 + xy + 2y^2 = 44, \\ 2x^2 - xy + y^2 = 16. \end{cases}$

III. *When each equation contains only one second degree power or product* (as x^2, y^2, or xy) *which is the same in all of the equations.* To solve such problems eliminate the second degree term and then proceed as in § 218.

1. $\begin{cases} 2y^2 + x = 30, \\ y^2 + 3x - 4y = 26. \end{cases}$ 2. $\begin{cases} x - \dfrac{1}{y} = a, \\ y - \dfrac{1}{x} = \dfrac{1}{a}. \end{cases}$

IV. *When the two equations are symmetrical* (§ 175) *in x and y.*

1. $\begin{cases} x^4 + y^4 = 17, \\ x + y = 3. \end{cases}$ SUGGESTION. Put $\begin{cases} x = u + v, \\ y = u - v. \end{cases}$

2. $\begin{cases} x^4 + y^4 = 82, \\ x - y = 2. \end{cases}$ 3. $\begin{cases} x^5 + y^5 = 33, \\ x + y = 3. \end{cases}$

V. *Solution first for* **function** *of the two unknowns, followed later by solution for the unknowns.*

1. $\begin{cases} (1)\ x^2 + 4y^2 - 15(x + 2y) + 80 = 0. \\ (2)\ xy = 6. \end{cases}$

Solution.

$$\begin{array}{lll}
(1_1) & x^2 + 4y^2 - 15(x+2y) + 80 = 0 & \\
(2_1) & \underline{4xy \qquad\qquad\qquad\qquad - 24 = 0} & \\
(3) & (x+2y)^2 - 15(x+2y) + 56 = 0 & (\text{Ax. ?})
\end{array}$$

We now solve for the *function* $x + 2y$, using factoring method.

$$\begin{array}{lll}
(3_1) & [x+2y-7][x+2y-8] = 0, & (\S\ 208) \\
(3_2) & x + 2y = 7, \text{ and } (3_3)\ x + 2y = 8. &
\end{array}$$

The solution is continued by combining (3_2) and (2), from which values of x and y are found. Then, in the same way, (3_3) and (2) are combined and other values of x and y found.

2. $\begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$

3. $\begin{cases} x^2 + 2xy + y + 3x = 73, \\ y^2 + 3y + x = 44. \end{cases}$

4. $\begin{cases} 9x^2 + y^2 - 63x + 21y = -86, \\ xy = 4. \end{cases}$

5. $\begin{cases} x^2 + y^2 + 4\sqrt{x^2 + y^2} = 45, \\ xy = 12. \end{cases}$

VI. *Equations containing three unknowns.*

1. $x^2 + y^2 = a;\ x^2 + z^2 = b;\ y^2 + z^2 = c.$

2. $\begin{cases} x^2 + y^2 + z^2 = 30, \\ xy + yz + zx = 17, \\ x - y - z = 2. \end{cases}$

Suggestion. Add 2 times (2) to (1) and extract square root of result, getting (4). Then combine (4) and (3).

3. $\begin{cases} x^2 + y^2 + z^2 = 84, \\ x + y + z = 14, \\ xy = 8. \end{cases}$

Suggestion. Add $2xy = 16$ to (1), and substitute $z = 14 - (x + y)$ in resulting equation. Then solve first for $x + y$.

VII. *Special devices applicable to particular problems.*

1. $\begin{cases} x^2 + xy = 144, \\ xy + y^2 = 180. \end{cases}$

Suggestion. Add the equations and extract square root getting (3). Then divide (1) by (3).

2. $\begin{cases} 3x^2 - xy + 3y^2 = 13, \\ x^2 - 3xy + y^2 = -1. \end{cases}$

Suggestion. Find value of xy. Then values of $x^2 \pm 2xy + y^2$.

3. $\begin{cases} x + xy + y = 5, \\ x^2 + xy + y^2 = 7. \end{cases}$

Suggestion. Add the equations and arrange to solve for $x + y$.

232. The General Case in Simultaneous Quadratics. Since all the problems so far undertaken have been solved, the student may get the impression that any problem in simultaneous quadratics can be solved. To show that such is not the case, we undertake to solve what appears to be a simple problem, much simpler than many we have had.

Given (1) $x^2 + y = a$, (2) $x + y^2 = b$, to find x and y.

From (2), $x = b - y^2$.

Then (1) $(b - y^2)^2 + y = a$,

or, $b^2 - 2by^2 + y^4 + y = a$.

Now this is an equation of the fourth degree. As a matter of fact the equation of the fourth degree has never been solved by any one in the sense of getting formulas for x and y from which values of x and y can be found by ordinary methods for any and every value of a and b.

An article in the American Mathematical Monthly, Vol. X, p. 192, deals with the question of the solubility of simultaneous quadratics. See Vol. VI, p. 13, for solutions of $x^2 + y = 7$, $x + y^2 = 11$.

233. Miscellaneous Problems in Quadratics.

1. The diagonal and longer side of a rectangle are together equal to 5 times the shorter side, and the longer exceeds the shorter by 35 m. Find the area of the rectangle.

2. The height s ft. to which a ball thrown vertically upward with a velocity of 100 ft. per second rises in t sec. is given by the formula $s = 100t - 16t^2$. Find t when $s = 144$.

a. The algebraic solution includes the two cases in which the ball is 144 ft. from the ground in the upward motion and 144 ft. from the ground in the downward motion. Verify this statement.

3. The plate of a mirror is 18 by 12 in., and it is to be framed with a frame of uniform width whose area is to be equal to that of the glass. Find width of frame.

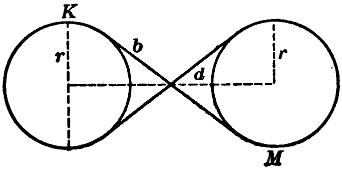
4. Two chords intersect in a circle. The segments of the first are represented by $7x + 2$ and $2x + 3$, and those of the other by $5x - 2$ and $4x - 1$. Find values of x to one decimal place and the values of the segments.

5. One side of a rectangle is $2x$ and diagonal is $2x+1$. If area is 12, find x and then diagonal.

6. If the median m on side c of a triangle is 12 in. and a and b are 11 in. and 15 in. respectively, find c from the formula $4m^2 = 2(a^2+b^2)-c^2$.

7. Find a formula for the volume v (in cu. ft.) of a sphere in terms of s, which represents its surface in sq. ft., knowing that $s=4\pi r^2$, and $v=\frac{4}{3}\pi r^3$, where r is radius (in feet).

8. The area of the ring included between two concentric circles is 84 sq. in. If the radius of the inner circle is 3.5 in., find the radius of the outer from the formula $a=\pi(R^2-r^2)$.



9. If the distance between centers of two equal pulleys of radius r ft. is d ft. and b ft. is the length of belt to go round them crossing so as to reverse the direction in which they turn, then $b=2\sqrt{4r^2+d^2}+2\pi r$ (approximately), KM being regarded as a straight line. Solve this equation for r.

10. A moving picture film, 120 ft. long, is made up of a certain number of small equal-sized pictures. If these pictures were .1 in. less in length on the film, there would be 720 more of them. How long is each small picture?

11. The diagonals of a rhombus are $2x+4$ and $2x-2$, and one side is $x+3$. Find x and then one side.

SUGGESTION. Observe what angle diagonals make with each other.

12. Show that it is impossible to find three consecutive integers such that the product of the first two equals the second minus the last.

SUGGESTION. Show that roots are imaginary.

13. Two trains, 50 mi. apart, on roads running at right angles to each other, are moving towards the crossing. One train runs 5 mi. an hour slower than the other. At what rates do they run if they meet at the end of 2 hr.?

14. The driving wheels of a locomotive are 2 ft. greater in diameter than the running wheels. The running wheels make 140 more turns in going a mile than the driving wheels. What are the diameters of the drivers if π is taken as $3\frac{1}{7}$?

15. Find to 2 decimal places the lower base b of a frustum of a cone whose volume v is 125 cu. in., altitude h is 5 in., and area of upper base b' is 20 sq. in., from the formula

$$v = \frac{1}{3} h (b + b' + \sqrt{bb'}).$$

234. Historical Notes. Though the Greeks were eminently successful in perfecting elementary geometry, very little progress was made by them in the study of either arithmetic or algebra. They studied arithmetic, strange to say, from the standpoint of geometry, representing numbers by the lengths of lines. A notation they had for numbers used the first ten letters in their alphabet to denote 1 to 10 ($a = 1$, $b = 2$, etc.), the next eight letters to denote multiples of 10 from 20 to 90, and the last nine letters to denote the hundreds. The fact that they used letters instead of special characters for the first ten numbers probably explains in part why they made little progress in algebra.

Diophantus, of Alexandria, Egypt, is recognized as the first writer on algebra worthy of the name. It is supposed he was of Greek descent, but he wrote about 350 A.D., long years after the Greek learning had developed. In the first part of his "*Arithmetica*" he solved determinate equations both simple and quadratic, but he did not accept negative or surd answers for the latter. Most of his book is given up to the solution of indeterminate equations of the second degree, in which analysis he showed great skill. The solution of such problems, however, even up to the present time, has not been reduced to a science, and, as a rule, no reference is made to them in elementary algebras. The biography of Diophantus is as brief or briefer than that of the great geometer Euclid: we know next to nothing concerning either of them. Diophantus's epitaph is often given as a problem in algebra.

Chronologically the next writer was Aryabhatta, who lived in India about 500 A.D. The arithmetic and algebra part of his book consisted merely of some rules written in verse. The next Hindu writer was Brahmagupta, who lived about a century later than Aryabhatta. His algebra, also written in verse, included the solution of equations, simple and quadratic, and some indeterminate problems, all written out in full in words. These writers used credits and debts to illustrate positive and negative numbers. The representing of numbers by means of the figures we have is supposed to have originated in India about this time. Without the Arabic notation calculations are largely performed with a wire frame, or abacus, or its equivalent. Thus, the Arabic notation is arithmetic without wires.

During the eighth century, by order of the Caliph Haroun Al-Raschid (the caliph of "The Arabian Nights"), the Hindu works on arithmetic and algebra were translated into Arabic, along with many other scientific books. In this way algebra came to be studied by a considerable number of scholars in Arabia. Prominent among the Arab writers on algebra was Al-Chwarizmi. He solved quadratic equations geometrically.

Thus, to solve $x^2+40x=329$ he let $x=$ one side of the square b^2 (see figure, § 63), and, extending its sides, made another square a^2 whose side was taken equal to 20 ($=\frac{1}{2}$ of 40). Then x^2+40x represented the area $b^2+ab+ab$. Adding the square a^2 to both sides of the equation, he had

$$x^2+40x+400=729.$$

Thus, he got for one side of the whole square $\sqrt{729}$ or 27. Subtracting 20, one side of a^2, from 27, he had the root sought, 7.

In the fourth part of his book Al-Chwarizmi attempts to prove the theorems $\sqrt{a^2b}=a\sqrt{b}$, and $\sqrt{a}\sqrt{b}=\sqrt{ab}$ (see §§ 191, 182), and in the last he gives some problems. This book is the more interesting because from it the Europeans got their knowledge of algebra and of the Arabic notation for numbers.

In the twelfth century the third Hindu writer on algebra, named Bhaskara, appeared. His algebra showed a great improvement over the earlier books, using abbreviations for words and being almost symbolic. In various ways he tried to sugar-coat his treatment to make it interesting. Thus, he gave the following problem:

The square root of half the number of bees in a swarm have flown out upon a jessamine bush, $\frac{8}{9}$ of the whole swarm have remained behind: one female bee flies about a male that is buzzing within a lotus-flower into which he was allured in the night by its sweet odor, but is now imprisoned in it. Tell me the number of bees. (*Ans.* 72.)

His arithmetic contained a clear statement of the so-called Arabic notation for numbers, including the 0. It took several hundred years for the world to learn the use and importance of 0 in the Arabic notation. The work of Bhaskara was known to the Arabs as soon as it was published.

Algebra and the Arabic notation for numbers got into Europe chiefly in this way. Leonardo Fibonacci was the son of an Italian merchant who represented his country in Barbary, and received his education there. He became acquainted with the Arabic system of numeration and with the Arabic works on algebra. On his return to Italy, 1202 A.D., he published his work "Liber Abaci," in which he set forth the advantages of the Arabic system over the old Roman notation with the letters I, V, X, etc., still in use among his countrymen. His treatment was taken from Al-Chwarizmi's and the other Arab works on the subject, and was written out in full in words. It had a wide circulation and helped materially in spreading a knowledge of algebra among Europeans.

Three or four centuries elapsed after the first introduction of algebra into Europe before European mathematicians took hold of the subject with the idea of perfecting it. In the sixteenth century two great results were accomplished: the first, the perfecting and establishing of the symbolic notation; and the other, the solving of the cubic equation. We have already learned how Vieta took a prominent part in introducing the symbolic notation. It remains to recount how the great mountain of the cubic equation was overcome.

In the discovery and publication of the solution of the cubic four persons took part. Of these Tartaglia is altogether the most interesting personality.

Tartaglia (real name Niccola Fontano) was born in Brescia, Italy, in the year 1500 A.D. When a child of twelve, Brescia was taken by the French and many of the inhabitants were massacred in the cathedral. The boy was in the cathedral and was left for dead, his skull having been split through in three places and his jaws and palate cut open. His mother found him, and, being deprived of other aid, remembered that dogs when injured always licked the wounds. To this care Tartaglia ascribed his recovery. His injury made him a stammerer, and in this way he got the name Tartaglia, which means stammerer. His mother taught him to read and write. Being highly gifted mathematically, before he was thirty-five years old he was chosen to the chair of mathematics at Venice. Later he became a recognized authority on the subject of gunnery, which was interesting the European world at that time.

Before this time, in 1505, **Scipione Ferro** had either discovered a way of solving cubics of the form $x^3 + mx = n$, or had found it in an Arab work. Ferro explained the method to his pupil **Fiori** (or Floridus). In those days it was not the custom to publish discoveries as now, but to keep them a secret so as to be able to solve in intellectual combat problems rivals could not solve, or so that students could not get the solution except in the school of the possessor of the secret.

About 1530 Tartaglia made known that he was in possession of a method for solving a cubic of the form $x^3 + px^2 = q$. Fiori, hearing of it, announced that he also had a method for solving difficult problems. Tartaglia then challenged Fiori to a public contest to take place Feb. 22, 1535, and, following the custom then in vogue, stakes were deposited with a notary with the understanding that whoever could solve the most problems out of a collection of thirty propounded by the other, was to get the stakes. Tartaglia suspected Fiori's problems would depend on the solution of cubic equations, and so he set to work to solve equations of the form $x^3 + mx = n$, and succeeded in accomplishing his

Tartaglia (1500–1559)

purpose just ten days before the date of the algebraical duel. When the contest took place, Tartaglia solved all of his opponent's problems in two hours, and by reason of his brilliant effort, won fame. By 1541 he had mastered the general solution of the cubic. (See § 230, III.)

Cardan was much interested in the contest between Tartaglia and Fiori, and as he had begun writing a book on mathematics, coaxed Tartaglia to explain his solution to him. Under the most solemn promises by Cardan to make no use of the knowledge, Tartaglia finally gave him the solution desired. But Cardan, breaking faith, some time after published the solution in his *Ars Magna,* the third earliest printed book on algebra. When this book appeared in 1545, Tartaglia was very angry, but the mischief had been done. Until very recently all algebras called the solution Cardan's.

Cardan, it seems, as soon as he learned the solution, taught it to his students, one of whom, **Ferrari**, succeeded in reducing the solution of the biquadratic, or equation of the fourth degree, to that of a cubic, and in this way solved the biquadratic.

After the discovery of how to solve cubic and biquadratic equations, mathematicians turned their attention to the solution of equations of still higher degrees, and all failed. Finally **Niels Henrik Abel** (1802–1829) of Norway succeeded in showing that the general solution of the fifth and higher degree equations cannot be expressed in terms of radicals.

There are many other things of interest that could be told of the history of algebra, but these the student will have to look up for himself in works on the subject, such as Ball's and Cajori's histories of mathematics published by Macmillan & Co., New York City.

CHAPTER IX

RATIO AND PROPORTION

I. PROPORTION PROPER

235. The Ratio of two quantities is obtained by dividing the first by the second. (See § 117.)

A **proportion** is an equation whose two members are ratios.

Proportions are written in two ways and read in two ways, both meaning the same thing. Thus,

either $a : b = c : d$,
or, $\frac{a}{b} = \frac{c}{d}$,

(The single colon denotes *division*. Formerly a double colon was used instead of =.)

may be read "a is to b as c is to d," or "the ratio of a to b equals the ratio of c to d."

236. Definitions. The first term of a ratio is called the **antecedent** and the last term the **consequent**. The first and last terms of a proportion are called the **extremes**, and the second and third terms the **means**. If the second and third terms of a proportion are the same quantity, this quantity is called a **mean proportional** between the first and last terms.

237. Fundamental Theorems about a Proportion.

1. *If four quantities are in proportion, the product of the extremes equals the product of the means.*

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. (By Mult. Ax.)

SUGGESTION. Both sides of the given equation are multiplied by bd, or it is cleared of fractions. See § 119.

2. *Conversely. If the product of two numbers equals the product of two other numbers, the factors of either product can be made the extremes, and the factors of the other the means of a proportion.*

If $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$. (By Div. Ax. Explain.)

238. Allowable Changes in the Order of the Terms of a Proportion.

1. **Alternation.** *If four numbers taken in order are in proportion, the ratio of the first to the third equals the ratio of the second to the fourth.*

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$. $\left(\text{Mult. Ax. Multiply through by } \frac{b}{c}.\right)$

2. **Inversion.** *If four numbers taken in order are in proportion, the ratio of the second to the first equals the ratio of the fourth to the third.*

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$. (Div. Ax. Divide 1 by each member.)

239. New Proportions from a Given One.

1. **Addition.** *If four numbers are in proportion, the ratio of the sum of the first and second to the first or second equals the ratio of the sum of the third and fourth to the third or fourth.*

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + 1 = \frac{c}{d} + 1$, or $\frac{a+b}{b} = \frac{c+d}{d}$. (Add. Ax.)

Also, $\frac{b}{a} = \frac{d}{c}$, whence $\frac{b}{a} + 1 = \frac{d}{c} + 1$, or $\frac{a+b}{a} = \frac{c+d}{c}$. (Add. Ax.)

2. **Subtraction.** *If four numbers are in proportion, the ratio of the difference of the first and second to the first or second equals the ratio of the difference of the third and fourth to the third or fourth.*

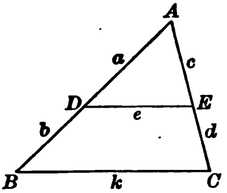
Proved in manner similar to addition.

a. **Addition** and **subtraction** have been called respectively "composition" and "division."

240. A Geometrical Meaning given to the Preceding Theorems. A theorem in geometry says, *If a line is drawn parallel to the base* *of a triangle, it divides the sides proportionally,* or $a:b=c:d$. By the preceding theorems we can now infer that the following proportions are also true:

$$\frac{AD}{AE}=\frac{BD}{EC};\ \frac{BD}{DA}=\frac{EC}{AE};$$

$$\frac{AB}{AD}=\frac{AC}{AE};\ \frac{AB}{BD}=\frac{AC}{EC}.$$



The two triangles ABC and ADE are similar, being mutually equiangular. Hence,

$$AB:AD=BC:DE.$$

We can see now that all these results can be summed up very simply in the single statement, *that the ratio of any two corresponding lines in the figure equals the ratio of any other two corresponding lines.*

241. Other Proportions from a Given One.

1. **Powers and Roots.** *If four numbers are in proportion, like powers or like roots of these terms are in proportion.*

If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^n}{b^n}=\frac{c^n}{d^n}$, and $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}$. (Power and Root Axs.)

2. **Continued Proportion.** *In a series of equal ratios, the ratio of the sum of the antecedents to the sum of the consequents equals any one of the ratios.*

If $\frac{a}{g}=\frac{c}{h}=\frac{e}{k}(=r)$, then $a=gr$, $c=hr$, $e=kr$, since in each case the dividend (numerator) equals the divisor times the quotient. Then, adding these equations,

$$a+c+e=gr+hr+kr=(g+h+k)r.\quad \text{(Add Ax.)}$$

$$\therefore\ \frac{a+c+e}{g+h+k}=r, \text{ or } \frac{a+c+e}{g+h+k}=\frac{a}{g}.\quad \text{(Div. Ax. and Ax. 8. § 57.)}$$

This result easily takes on a geometrical meaning by reference to the figure of the preceding article.

Let $e=DE$, $g=AB$, $h=AC$, $k=BC$.

Then, we have, ***the ratio of the perimeter of triangle ADE to the perimeter of triangle ABC equals the ratio of any two corresponding sides.***

By using a letter for each side whatever the number of sides a polygon has, this theorem is easily extended to the case of *any* similar polygons. In fact, this theorem appears in algebras mainly because it has this application in geometry.

242. Exercise in Proportion.

1. If $y:x=7:2$, find what the ratio $x:y$ equals. See § 238, 2.

2. From $12\,x=18\,y$ find the ratio of x to y. See § 237, 2.

3. From $4x-9y=2x+5y$ find the ratio of x to y.

4. From $3x+y:y=17:8$ find the ratio of x to y. (§ 237, 1.)

5. If the ratio of m to n is $\frac{7}{4}$, what is the ratio of $m+n$ to $m-n$? (§ 239.)

6. Solve $x^2-4:x^2-9=x^2-5x+6:x^2+4x+3$. (§ 237, 1.)

7. If $a:b=b:c$, prove that $a+b:b+c=a:b$.
(239, 1, and § 238, 1.)

8. What number must be added to each term of the ratio $m:n$ so that the new ratio may equal the ratio of $p:q$? (Let $x=$ the number.)

9. Find a mean proportional between $a^2+2ab+b^2$ and $a^2-2\,ab+b^2$. (§ 236.) (Let $x=$ mean proportional.)

10. Find the fourth term of the proportion whose first three terms are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

11. Find x from the proportion

$$6x+a:4\,x+b=3x-b:2x-a.$$

12. Four given numbers are represented by m, n, p, q. What number added to each will give four numbers that are in proportion? (Let $x=$ the number added to each.)

13. Find two numbers whose sum, difference, and product are proportional to m, n, and p. (Let x and y be the numbers. See § 241, 2, where a, c, and e are proportional to b, d, and f.)

14. Find two numbers such that their sum is to their difference as $5:1$, and their sum is to their product as $5:4$.

15. Prove that if the terms of $a:b=c:d$ are divided by the respective terms of $a':b'=c':d'$, the quotients are in proportion.

16. If $a:b=c:d$, prove that $\frac{2a+3c}{2a-3c}=\frac{8b+12d}{8b-12d}$.

SUGGESTION. First alternate the given proportion, then multiply the first ratio by $\frac{2}{3}$, and the second ratio by $\frac{8}{12}$, then apply "addition" and "subtraction" to both ratios in turn, and finally divide the terms of one proportion by the corresponding terms of the other.

17. If $a:b=c:d$, prove that $a^2+b^2:a^2-b^2=c^2+d^2:c^2-d^2$.

18. If $a:b=c:d$, prove that $a+b:c+d=\sqrt{a^2+b^2}:\sqrt{c^2+d^2}$.

SUGGESTION. Take first $a:b=c:d$ and $a^2:b^2=c^2:d^2$ by "addition."

19. Prove that either root of $x^2-q=0$ is a mean proportional between the roots of $x^2+px+q=0$.

20. If $a:b=c:d$ show that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2.

SUGGESTION. State the proportion, and then multiply means and extremes together. Show that this equation reduces to the given proportion, if $a:b=c:d$. Then reverse the steps.

21. What is the ratio of the mean proportional between a and b to the mean proportional between a and d? Give this in its simplest form.

22. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, prove that $a:b=c:d$.

23. If $a:b=4:5$, $d:f=5:2$, $e:c=6:7$, $d:b=7:3$, and $f:c=4:3$, $a:b:c:d:e:f=$ what continued ratio of numbers?

24. The product of two numbers is 112, and the difference of their cubes is to the cube of their difference as $31:3$. What are the numbers?

II. VARIATION

243. Variation. One quantity is said to vary directly as another, when the two are so related that the ratio of any two values of the one is equal to the ratio of the corresponding values of the other. The special symbol for variation is $\propto$, read "varies as."

Direct Variation. In ordinary scales where a weight is moved from 0 along a scale beam until the scale balances, the weight of an object w varies directly as the distance d the scale weight is moved, or $w \propto d$. Thus, if the scale weight is moved from 0 twice as far one time as another, the first object is twice as heavy as the other. The circumference c of a circle varies as its diameter d, or $c \propto d$. The graph for direct variation is a straight line. See § 122.

Inverse Variation. The daily wages w a man gets for doing a job for a fixed sum varies inversely as the number of days n he works, or $w \propto \frac{1}{n}$. The *greater* the number of days he works, the *less* money he gets per day. The number of revolutions n the driving wheels of a locomotive make in going a mile varies inversely as the diameter d of the wheels, or $n \propto \frac{1}{d}$.

Other Forms of Variation. The distance s through which a falling body moves varies as the square of the time t it is moving, or $s \propto t^2$. Thus, if the time is trebled, the distance is multiplied by 9. Again, the power of attraction a of the heavenly bodies for one another varies inversely as the square of their distance apart d, or

$$a \propto \frac{1}{d^2}.$$

In any problem in variation the sign $\propto$ can be replaced by an equality sign by introducing a constant factor c before the right member. Thus, $w \propto d$ becomes $w = cd$; $w \propto \frac{1}{n}$ becomes $w = \frac{c}{n}$; etc.

In any given physical problem c has to be found by experiment to fit the units of distance, weight, time, etc. Thus, in $s = ct^2$, if distance is measured in feet, time in seconds, and gravity is the force, $c = \frac{1}{2} g$, in which $g = 32.16$.

The graph for inverse variation is the hyperbola, since $w \cdot n = c$, where c is constant. See § 221, 4 and *e*.

244. Variation a Form of Proportion. To show that variation is a form of proportion, take the case of the diameter of a circle and its circumference.

Thus, let d_1 and d_2 be the diameters, and c_1 and c_2 the circumferences of any two circles. Then,

$$\frac{c_1}{d_1} = \frac{c_2}{d_2}.$$

(*By theorem: The ratio of the circumference to the diameter is the same for all circles.*)

By alternation (§ 238, 1), it follows that

$$\frac{c_1}{c_2} = \frac{d_1}{d_2}.$$

Thus, the ratio of any two circumferences equals the ratio of their diameters. This is precisely what is meant by saying the circumferences of circles vary as their diameters, or that $c \propto d$. If the circumference of a circle is doubled or trebled, its diameter likewise is doubled or trebled.

Again, since the area of a circle equals π times the square of its radius, we can say that the areas of circles vary as the squares of their radii, or $a \propto r^2$.

To show this let a_1 and a_2 be the areas of two circles, and r_1 and r_2 their respective radii. Then,

$$\frac{a_1}{a_2} = \frac{\pi r_1^2}{\pi r_2^2} = \frac{r_1^2}{r_2^2}. \quad \text{(By dividing both terms by } \pi, \text{ § 98.)}$$

Since a problem in variation is a problem in proportion, three terms are always given to find the fourth. Two of these terms give a ratio, whose value is the constant c referred to in § 243.

To solve a problem in variation, (1) *First replace the symbol* $\propto$ *by* $=$ *at the same time introducing the constant factor* c *before the right member.*

(2) *Then substitute two of the given quantities which correspond in the equation found in* (1), *getting the value of* c *therefrom. Substitute this value of* c *and the remaining given quantity in the equation of* (1) *to find the fourth or required term.*

245. Exercise in solving Problems in Variation.

1. The weight W of a cube varies as the cube of its edge e, or $W \propto e^3$. If when W is 26 lb. e is 4 in., what is W when e is 5 in.?

SOLUTION. $W = ce^3$. Then $26 = c \times 64$, whence $c = \frac{13}{32}$. Then $W = \frac{13}{32} \times 125 = 50.8^-$ lb.

2. The cost C of plastering a wall varies as the product of its length l and height h, or $C \propto lh$. If it costs \$11.70 to plaster a wall 21 ft. long by 12 ft. high, what will it cost to plaster a wall 37 ft. long by 11 ft. high?

3. The distance s fallen by a body from rest varies as the square of the time t during which it falls. If owing to the resistance of the atmosphere it falls only 560 ft. in 6 sec., what distance will it fall in 5 sec.?

NOTE. When the resistance of the atmosphere is taken into account, the law holds only approximately, but the error will be slight.

4. The illumination i of a book by a lamp varies inversely as the square of the distance d of the book from the light. If the illumination at 18 in. is 48 candle power, what quantity of light will fall on the book at 4 ft.?

5. If surface of sphere is given by formula $s = 4\pi r^2$, how does s vary as r changes? *Ans.* s varies as square of r.

6. If $v = \frac{4}{3}\pi r^3$ is formula for volume of a sphere, how does v vary with r? If r is doubled, what happens to v? How does r vary with v? *Ans.* As the cube root of v.

7. In $t = 2\pi\sqrt{\frac{l}{g}}$ how does t vary with l? With g?

Ans. to last. Inversely as the square root of g.

8. If T is tension on a string, W the weight of a unit length of it, l its length, and N the number of its vibrations per second, tell from the formula

$$N : N' = \frac{\sqrt{T}}{l\sqrt{W}} : \frac{\sqrt{T'}}{l'\sqrt{W'}}$$

how N varies with T; with W; with l.

246. Variation and Functionality. In §§ 123–126 we developed the idea of a function gradually changing in value as the quantity on which it depends changes. We there illustrated this change by means of graphs.

We can now throw some additional light on this subject from our brief study of variation. We see in every one of the formulas given, how one quantity is dependent for its value on another quantity in the formula, and how one changes continuously as the other changes continuously.

Thus, the quantity of light falling on a page of a book varies inversely as the square of the distance from the light. As the book is moved away from the light the quantity of light falling on the book continuously decreases rapidly.

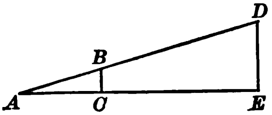
The lifting force of a simple lever varies directly with the power and also with the length of the power arm, and inversely with the length of the weight arm.

The horse power of an engine varies directly as the steam or gas pressure; it varies also directly as the length of the stroke of the piston; also directly as the area of the cylinder; also directly as the number of strokes per second.

It must be plain to the student from these illustrations that the range of application of the idea of variation or functionality is as wide as science itself, whether theoretical or applied. One branch of higher mathematics, called the calculus, deals largely with the continuous changes in values of functions.

III. THE RATIOS OF THE SIDES IN RIGHT TRIANGLES*

247. Similar Triangles. Similar triangles are defined in geometry as those having their corresponding angles equal and their corresponding sides proportional.



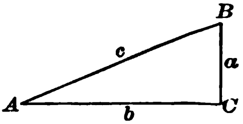
Now two right triangles are similar if they have an acute angle of one equal to an acute angle of the other (§ 51).

Hence, if ABC and ADE are right triangles having the acute angle A common,

$$\frac{BC}{AB}=\frac{DE}{AD};\quad \frac{AC}{AB}=\frac{AE}{AD};\quad \frac{BC}{AC}=\frac{DE}{AE}.$$

Thus, we see that so long as two right triangles have the same-sized acute angle A, no matter how much larger one triangle is than the other, the ratio of any two sides of one triangle is equal to the ratio of the corresponding sides of the other triangle.

248. The Ratios of the Sides of a right triangle to one another are called the **trigonometrical functions** of its acute angles. These functions have been given distinctive names.



Let A be an acute angle of a right-angled triangle, with the sides and angles marked as in the diagram, a being opposite angle A, and b opposite angle B. There are six ratios that can be formed out of the three sides, viz.:

$\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{b}{a}, \frac{c}{b}, \frac{c}{a}$. Of the last two, called "secant" (sec) and "cosecant" (cosec), we will make no use.

Calling the others "sine," "cosine," "tangent," "cotangent," we now write

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* The remainder of this chapter can be omitted if desired. Several reasons can be given for taking it up here if there is time. It deals with practical problems, some appearing in physics, is a preparation for learning to use the logarithmic table, and gives a good start in trigonometry. Probably the chief difficulties experienced by students in its study are algebraic in character.

(1) Sine $A = \frac{a}{c}$; or, sine $A = \frac{\text{side opposite to } A}{\text{hypotenuse}}$.

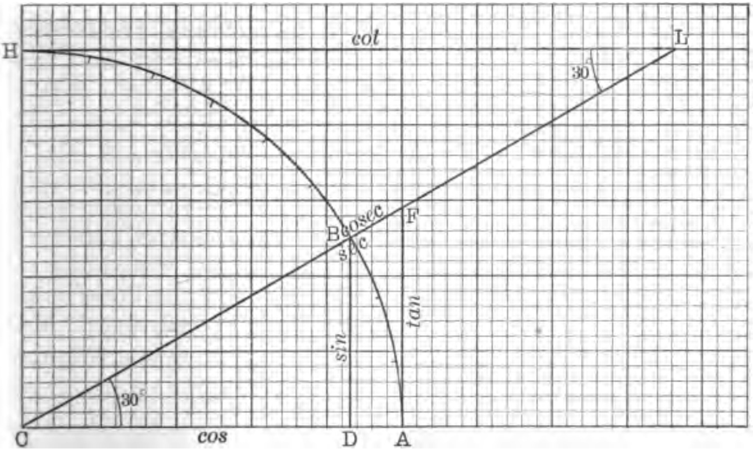
(2) Cosine $A = \frac{b}{c}$; or, cosine $A = \frac{\text{side adjacent to } A}{\text{hypotenuse}}$.

(3) Tangent $A = \frac{a}{b}$; or, tangent $A = \frac{\text{side opposite to } A}{\text{side adjacent to } A}$.

(4) Cotangent $A = \frac{b}{a}$; or, cotangent $A = \frac{\text{side adjacent to } A}{\text{side opposite to } A}$.

a. The prefix "co" in cosine and cotangent, means "complement," so that "cosine" means "complement sine," that is, "sine of complement." Notice that B (which equals $90° - A$) is the *complement* of A. Also that sine $B = \frac{b}{c} \left(\frac{\text{side opposite } B}{\text{hypotenuse}}\right) =$ cosine A; and that cosine $B = \frac{a}{c} =$ sine A.

249. Construction of a Table of Trigonometrical Functions for every 10° of angle from 0° to 90°.



We construct a quadrant of a circle (see diagram) with the lower left-hand corner of a sheet of squared paper as center and having a radius of 5 cm. We then divide the quadrant arc, with a protractor or compass, into nine equal parts of 10° each, and construct a straight line from the center through the 30° point of division of the arc, producing it until it meets the two tangent lines AF and HL. By reading directly from the squared paper, we can tell the length of all vertical distances, such as BD and AF, and of all horizontal lines, such as CD and HL, these lines belonging to the angle of 30°, as shown on the diagram. Notice that $CA = CB = CH = 5$ cm.

Now by the definitions in § 248 we have

$$\sin 30^\circ = \frac{BD}{BC} = \frac{2.5 \text{ c.m.}}{5 \text{ cm.}} = .50.$$

$$\cos 30^\circ = \frac{CD}{BC} = \frac{4.3 \text{ cm.}}{5 \text{ cm.}} = .86.$$

$$\tan 30^\circ = \frac{AF}{AC} = \frac{2.9 \text{ cm.}}{5 \text{ cm.}} = .58.$$

In getting cot 30°, to avoid dividing AC, 5 cm., by AF, 2.9 cm., we use triangle CHL, in which angle $L = 30^\circ$.

$$\cot 30^\circ = \frac{HL}{HC} = \frac{8.6^+ \text{ cm.}}{5 \text{ cm.}} = 1.7^+.$$

The student may now make a diagram like that just given, but whose radius is 10 cm., and construct lines not merely through the center and the 30° point, but also through the center and *all* the points of division of the arc, extending them until they meet the two tangents AF and HL (which lines will have to be made longer than in the diagram given in the book). From this diagram he may fill out the blank spaces in the following Natural Function table by getting the corresponding measurements and quotients for the angles 0°, 10°, 20°, etc. Notice that the lines corresponding to BD and AF for 0° are each 0, and that the lines corresponding to HL when the angle is 0° is infinite in length (§ 9), as also the line corresponding to AF when the angle is 90°.

NATURAL FUNCTION TABLE

EXPLANATION. When filling in the values for the degrees at the left side of the table, read the functions at the top, but when filling in the values for the degrees at the right of the table, read the functions at the bottom. It will be found that

sin 10° = cos 80° (§ 248, *a*),
tan 30° = cot 60°,
etc.

	SIN	TAN	COT	COS	
0°	0	0	∞	1.00	90°
10°					80°
20°					70°
30°	.50	.58	1.7	.86	60°
40°					50°
	COS	COT	TAN	SIN	

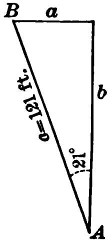
The student may now compare the values he has obtained with those given in the Table of Natural Functions found on page 224.

a. It is important to note that each of the expressions "sin A," or "cos 20°," or the like, denotes *one number*, and has to be treated as such in making algebraical transformations. In algebra each letter in "sin A" would denote a number, and sin A would signify their product; but in trigonometry these four letters are handled *as if they were a single letter.*

250. Use of the Table of Functions. If we know the values of certain "parts" (sides or angles) of any right triangle, we can find the remaining parts by means of the trigonometrical ratios to be found in the table, page 224, for all the different sizes of angles 0°–90°. Examples will make this plain.

251. Exercise in Solving Problems using the Four-Place Natural Function Table (page 224).

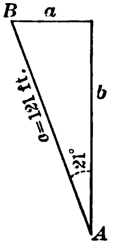
1. A man walks 121 ft. in the direction 21° west of north. How far west has he gone? How far north?



SOLUTION. (1) To find a, the distance directly west.

We have $\sin A = \frac{a}{c}$ (§ 248); $a = c \times \sin A$. (Mult. Ax.)

Angle	Sine	Tangent	Cotangent	Cosine	
0°	0	0	∞	1	90°
1	.0175	.0175	57.2900	.9998	89
2	.0349	.0349	28.6363	.9994	88
3	.0523	.0524	19.0811	.9986	87
4	.0698	.0699	14.3006	.9976	86
5	.0872	.0875	11.4301	.9962	85
6	.1045	.1051	9.5144	.9945	84
7	.1219	.1228	8.1443	.9925	83
8	.1392	.1405	7.1154	.9903	82
9	.1564	.1584	6.3138	.9877	81
10	.1736	.1763	5.6713	.9848	80
11	.1908	.1944	5.1446	.9816	79
12	.2079	.2126	4.7046	.9781	78
13	.2250	.2309	4.3315	.9744	77
14	.2419	.2493	4.0108	.9703	76
15	.2588	.2679	3.7321	.9659	75
16	.2756	.2867	3.4874	.9613	74
17	.2924	.3057	3.2709	.9563	73
18	.3090	.3249	3.0777	.9511	72
19	.3256	.3443	2.9042	.9455	71
20	.3420	.3640	2.7475	.9397	70
21	.3584	.3839	2.6051	.9336	69
22	.3746	.4040	2.4751	.9272	68
23	.3907	.4245	2.3559	.9205	67
24	.4067	.4452	2.2460	.9135	66
25	.4226	.4663	2.1445	.9063	65
26	.4384	.4877	2.0503	.8988	64
27	.4540	.5095	1.9626	.8910	63
28	.4695	.5317	1.8807	.8829	62
29	.4848	.5543	1.8040	.8746	61
30	.5000	.5774	1.7321	.8660	60
31	.5150	.6009	1.6643	.8572	59
32	.5299	.6249	1.6003	.8480	58
33	.5446	.6494	1.5399	.8387	57
34	.5592	.6745	1.4826	.8290	56
35	.5736	.7002	1.4281	.8192	55
36	.5878	.7265	1.3764	.8090	54
37	.6018	.7536	1.3270	.7986	53
38	.6157	.7813	1.2799	.7880	52
39	.6293	.8098	1.2349	.7771	51
40	.6428	.8391	1.1918	.7660	50
41	.6561	.8693	1.1504	.7547	49
42	.6691	.9004	1.1106	.7431	48
43	.6820	.9325	1.0724	.7314	47
44	.6947	.9657	1.0355	.7193	46
45	.7071	1.0000	1.0000	.7071	45
	Cosine	Cotangent	Tangent	Sine	Angle



Turning to the Table on opposite page, for the value of sin A, and substituting it in the last equation, we get

$$a = 121 \times .3584 = 43.37^{-} \text{ ft.} \quad \textit{Ans.}$$

(2) To find B. $B = 90° - A = 90° - 21° = 69°$.

(3) To find b, the distance directly north. We have $\sin B = \frac{b}{c}$ (§ 248). Hence, $b = c \times \sin B$. (Mult. Ax.)

Then referring to the table for the value of sin 69° (and remembering that when the angle is over 45°, the function name must be read at the bottom of the page), we get

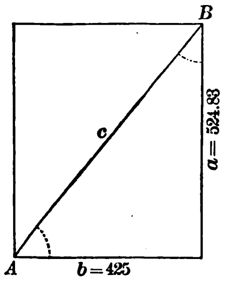
$$b = 121 \times .9336 = 112.97^{-} \text{ ft.} \quad \textit{Ans.}$$

2. Given $A = 32°$, $c = 259$ yd., to find a, B, and b.

3. Given $A = 42°$, $c = 12.4$ mi., to find a, B, and b.

4. Given $A = 73°$, $c = 2542$ m., to find a, B, and b.

5. Given the lengths of two sides of a rectangular block as 425 ft. and 524 ft. 10 in., to find the two angles this diagonal makes with the sides, and the length of the diagonal from one corner to the opposite one.



Solution. — (1) To find A, the angle between the diagonal and a side. We have

$$\tan A = \frac{a}{b} = \frac{524.83}{425} = 1.2349.$$

Turning to the Table, we find if $\tan A = 1.2349$ that $A = 51°$. *Ans.*

(2) To find B. $B = 90° - 51° = 39°$.

(3) To find the diagonal c.

$$\sin A = \frac{a}{c}. \text{ Hence, } c \times \sin A = a, \text{ and } c = \frac{a}{\sin A}. \quad \text{(Axs. ?)}$$

Substituting the value of sin A found in the table, we have

$$c = \frac{524.83}{.7771} = 675.4 \text{ ft.} \quad \textit{Ans.}$$

a. This answer can be checked by geometry, using the formula $c^2 = a^2 + b^2$. Notice that c can be found with less work by trigonometry than by geometry.

6. Given $a = 36.506$, $b = 82$, to find A, B, and c.

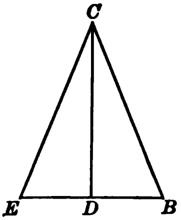
Find the remaining parts with the following conditions given.

7. $a = 40.394$, $b = 190$.

8. $a = 90.9$, $b = 225$.

9. $a = 63.028$, $b = 70$.

10. Given one side 18 ft. of a regular octagon (eight-sided equilateral and equiangular polygon), to find the perpendicular distance from its center to one side, and the distance from center to a vertex.



Solution. In the figure, C is the center of the polygon, EB is one side, and CD is the perpendicular from the center on the side. Then DCB is a right triangle in which we will regard $\angle DCB$ as angle A of our formula, DB as side a, CD as side b, and CB as side c. Then, $A = \frac{1}{16}$ of 360°, or $22\frac{1}{2}°$, and the opposite side $a = 9$ ft.

(1) To find b. We have,

$$\tan A = \frac{a}{b}; \text{ therefore } b \times \tan A = a, \text{ whence } b = \frac{a}{\tan A} = \frac{9}{.4143} = 21.72.$$

Explanation. The .4143, which is the tangent of $22\frac{1}{2}°$, is found by "interpolating." From the table,

$$\tan 22° = .4040, \ \tan 23° = .4245;$$

$$\tan 22\tfrac{1}{2}° = .4040 + \tfrac{1}{2}\,(.4245 - .4040) = .4143 \text{ (to nearest .0001).}$$

(2) To find c. We write

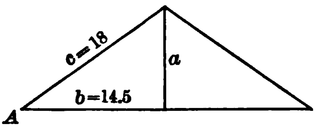
$$\sin A = \frac{a}{c}; \ \therefore c = \frac{a}{\sin A} \text{ (Axs.?)} = \frac{9}{.3827} = 23.52.$$

11. Given $A = 52\frac{1}{3}°$, $a = 35$ inches, to find b, c, and B.

Find the remaining parts in the following three problems, as in Ex. 10.

12. $A = 33\frac{3}{4}°$, $a = 25$. **13.** $B = 39°$, $a = 72$. **14.** $A = 48\frac{1}{4}°$, $b = 17.6$.

15. A house 29 ft. wide has a roof which measures 18 ft. from the ridge to either side of the house. Find the angle the roof makes with the horizontal, and the height of the ridge above the top story.



Solution. (1) $\text{Cos}\, A = \frac{b}{c} = \frac{14.5}{18} = .8056.$

Now, .8056 lies between cos 36° = .8090 and cos 37° = .7986. The difference for 1° is 104, and the difference between 8090 and 8056 is 34. Hence, the number of *minutes* of angle to be added to 36° is $\frac{34}{104}$ of $60'$, or $20'^-$. Therefore $A = 36^\circ\ 20'^-$. *Ans.*

(2) $\sin A = \frac{a}{c}$; $\therefore a = c \times \sin A = 18 \times .5925 = 10.67$ ft. *Ans.*

Explanation. To find the sine of $36^\circ\ 20'$, one gets $\sin 36^\circ$ and adds to it $\frac{20}{60}$ of the (tabular) difference between $\sin 36^\circ$ and $\sin 37^\circ$.

Important Note. The student should see that while the sine and tangent *increase* as the angle increases, though not in proportion to the increase in the angle, the cosine and cotangent *decrease* as the angle increases. Thus,

$\sin 0^\circ = .0000$	$\tan 10^\circ = .1763$	$\cos 20^\circ = .9397$	$\cot 20^\circ = 2.7475$
$\sin 30^\circ = .5000$	$\tan 30^\circ = .5774$	$\cos 40^\circ = .7660$	$\cot 40^\circ = 1.1918$
$\sin 60^\circ = .8660$	$\tan 60^\circ = 1.7321$	$\cos 60^\circ = .5000$	$\cot 60^\circ = .5774$
$\sin 90^\circ = 1.0000$	$\tan 80^\circ = 5.6713$	$\cos 80^\circ = .1736$	$\cot 80^\circ = .1763$

252. Rule for finding a Function from its Angle in the Natural Table, p. 224.

Look first for the function corresponding to the given number of degrees. Then, if there are minutes, get the "tabular difference" by subtracting the number found from that for the next higher number of degrees, and multiply this difference by the fraction of a degree denoted by the given number of minutes. If the function is a sine or a tangent, **add** *the product to the first number found in the table; but if it is a cosine or a cotangent,* **subtract** *it from that number.*

253. Rule for finding an Angle containing Minutes from its Function in the Natural Table.

First find the numbers in the table between which the given function lies, and subtract the less from the greater for the "tabular difference" for the corresponding interval. If the function is a sine or a tangent, subtract the smaller of the two numbers in the table from the given value of the function and take such a part of 60′ as the difference found is of the tabular difference, and add this number of minutes to the number of degrees corresponding to the lesser tabular number; but if the function is a cosine or a cotangent, subtract the given value of the function from the larger tabular number and take such a part of 60′ as the difference found is of the tabular difference, *and add this number of minutes to the number of degrees corresponding to the larger of the two tabular numbers.*

254. Rule for solving Right Triangles for Unknown Parts. In every problem two parts will always be given to find the remaining three. Of these, one, an acute angle, is always found by subtracting the other acute angle from 90°. To find the remaining parts two formulas will be needed.

To provide the formula for any given case, select that one of the formulas, (1), (2), (3), § 248, which contains the two given parts and the one that is to be found, and solve by algebra for the unknown quantity.

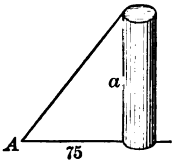
255. Exercise in solving for Unknown Parts in Right Triangles.

Find the remaining parts in the following nine problems:

1. $A = 25° 40'$, $b = 84$.
2. $A = 25° 35'$, $c = 25.6$.
3. $b = 73$, $c = 99$.
4. $b = 1291$, $c = 1674$.
5. $a = 43.1$, $c = 56.19$.
6. $a = 29.7$, $B = 73° 23'$.
7. $B = 24° 54'$, $a = 78.44$.
8. $a = 192.5$, $b = 173.8$.
9. $a = 19.65$, $b = 27.84$.
10. $B = 17° 18'$, $c = 84.7$.

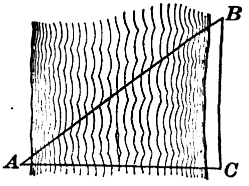
11. How high is a tower that casts a shadow 75 ft. in length, when the "angle of elevation" of the sun is 52° 35′?

SUGGESTION. $A = 52° 35'$, $b = 75$; to find a.



12. A flagstaff 90 ft. high casts a shadow 117 ft. long. Find the "altitude" of the sun, that is, the angle the line from the sun through the top of the staff makes with a horizontal line where it meets the ground. The altitude angle corresponds to angle of elevation A in the preceding exercise.

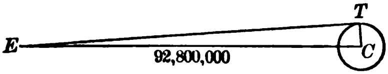
13. A distance BC is measured 124 ft. along the bank of a river from a point C opposite a tree at A on the other bank (ACB = right angle), and angle B is



measured and found to be 52° 24′. What does *AC*, the distance across the river, equal?

14. One side of a regular heptagon (seven-sided polygon) is 17 in. Find first the perpendicular on a side, and then the area of the heptagon. (See Ex. 10, § 251.)

15. The top of a lighthouse is 200 ft. above sea level, and the angle of depression to a buoy is 9° 53′. Find the horizontal distance of the buoy from the lighthouse.



16. The angle *E* at the earth's center subtended by the sun's radius is approximately 16′, and the sun's distance is 92,800,000 mi. Find the sun's radius in miles.

17. What must be the length of a ladder set at an angle of 71° 14′ with the ground to reach a window 22 ft. high?

18. The base of an isosceles triangle is 15 in., and the angle at the vertex is $46\frac{1}{2}°$. Find altitude of triangle.

19. The chord of a circle is 25 meters long, and the angle at the center subtended by it is 40° 15′. Find radius of circle.

20. Find radius of small circle in latitude 30° if it equals 3963 mi. × cos 30°. (See figure in § 249.)

21. Two forces act at *A* (figure in Ex. 5, § 251), the horizontal one *b* with force of 15 lb. and vertical one *a* with force of 19 lb. Find resultant force *c* and angle it makes with horizontal.

22. A river 1 mi. wide flows at the rate of 4 mi. an hour. A steam ferryboat with a speed of 8 mi. an hour has to run directly across. In what direction must she steer, and what time will she be in crossing?

23. The perimeter of a right triangle is 6.37 ft. and one of its acute angles is 33° 24′. Find the three sides.

CHAPTER X

LOGARITHMS

I. NATURE OF LOGARITHMS

256. Calculations made through Use of Exponents. Whenever numbers which are the powers of one number, as, for instance, of 2, are to be multiplied, divided, raised to powers, or have their roots extracted, these operations can be performed very quickly, if a table containing its various powers has been prepared.

TABLE

Number	Power of 2
2	2
4	2^2
8	2^3
16	2^4
32	2^5
64	2^6
128	2^7
256	2^8
512	2^9
1,024	2^{10}
2,048	2^{11}
4,096	2^{12}
8,192	2^{13}
16,384	2^{14}
32,768	2^{15}
65,536	2^{16}

1. Calculate 16×64 by use of the table.

Solution. $16 = 2^4$. (From the table.)

$64 = 2^6$. (From the table.)

$\therefore 16 \times 64 = 2^4 \times 2^6 = 2^{10}$. (Law of exponents.)

But $2^{10} = 1024$. (From the table.)

$\therefore 16 \times 64 = 1024$. *Ans.*

2. Calculate $8 \times 128 \times 16$ by using the table.

3. Calculate $16{,}384 \div 256$ with table.

Solution. $16384 = 2^{14}$. (From the table.)

$256 = 2^8$. (From the table.)

$\therefore 16384 \div 256 = 2^{14} \div 2^8 = 2^6 = 64$. (From the table.)

4. $\dfrac{64 \times 256 \times 16}{32 \times 512}$. 5. $\dfrac{8192 \times 512 \times 65536}{4096 \times 4 \times 32768}$.

6. Square 32 by use of table; also 256.

7. Extract square root of 16,384; of 4096.

8. Extract cube root of 32,768; of 4096.

9. Extract 4th root of 4096.

a. The examples just given show that calculations involving only numbers which are powers of 2 can be made very easily and expeditiously. Now, we can express *all* numbers as powers of 2 by the simple device of using *fractional exponents.* By calculations it is found that $15 = 2^{3.9}$; $144 = 2^{7.17}$. Similar expressions can be found for *any* number. Thus, we see how a table could be made with 2 for *base* which could be used to make calculations with any numbers. Moreover, some other number, as 3, could be used instead of 2 as the base in the table, all numbers being represented as powers of 3. As we shall see later, the number 10 is the base used in most tables of this kind.

257. Logarithms are Exponents. In $3^2 = 9$, we can say 2 is the logarithm of 9 to base 3. Similarly, since $3^4 = 81$, 4 is the logarithm of 81 to base 3. In logarithms three different numbers are always involved: (1) A number. (2) Its logarithm. (3) The base used.

The **logarithm of a given number** is the exponent of the power to which a base must be raised to produce this number.

A system of logarithms is a set of numbers with their logarithms all taken to the same base. Notice that the logarithm of 1 *in any system* is 0, since $a^0 = 1$ (§ 185).

SYSTEM OF LOGARITHMS WITH BASE 2

NUMBER	LOGARITHM	REASON	NUMBER	LOGARITHM	REASON
1	0	$2^0 = 1$			
2	1	$2 = 2$	$\frac{1}{2}$	-1	$2^{-1} = \frac{1}{2}$
3	1.5850	$2^{1.5850} = 3$	$\frac{1}{3}$	-1.5850	$2^{-1.5850} = \frac{1}{3}$
4	2	$2^2 = 4$	$\frac{1}{4}$	-2	$2^{-2} = \frac{1}{4}$
5	2.3223	$2^{2.3223} = 5$	$\frac{1}{5}$	-2.3223	$2^{-2.3223} = \frac{1}{5}$
8	3	$2^3 = 8$	$\frac{1}{8}$	-3	$2^{-3} = \frac{1}{8}$
etc.	etc.	etc.	etc.	etc.	etc.

The student is not supposed to know how the decimal logarithms, like 1.5850, are found. Originally they were obtained by a long process of extracting roots. (See the article "Logarithms" in the Encyclopædia Britannica.) Since logarithms are exponents, they may be interpreted as such. Thus, in the equation $2^{\frac{15850}{10000}} = 3$, we see that the 15850th power of the 10000 root of 2 equals 3, and if these operations were actually performed on 2, the result would be 3.

258. Notation and Terms. To avoid writing long exponents, such an equation as $2^{1.5850} = 3$ is changed into $\log_2 3 = 1.5850$, and is read "logarithm of 3 to base 2 equals 1.5850." The subscript indicating the base is usually omitted when 10 is the base.

The integral part of a logarithm is called its **characteristic**, and the decimal part its **mantissa**. Thus, in 1.5850, which is the logarithm of 3 to base 2, the 1 is the characteristic and the .5850 is the mantissa of the logarithm.

Express the following in the language of logarithms:

1. $2^4 = 16$.

SOLUTION. $\log_2 16 = 4$. Read 'the logarithm of 16 to base 2 is 4.'

2. $3^3 = 27$. 3. $10^3 = 1000$. 4. $4^2 = 16$. 5. $2^5 = 32$.

6. $3^{-2} = \frac{1}{9}$. 7. $10^2 = 100$. 8. $10^{-1} = \frac{1}{10}$. 9. $10^{-4} = .0001$.

Express the following equations, using the language of exponents:

10. $\log_2 8 = 3$. 11. $\log_4 64 = 3$. 12. $\log_5 25 = 2$.

13. $\log_8 4 = \frac{2}{3}$. 14. $\log_{10} .01 = -2$. 15. $\log_9 27 = \frac{3}{2}$.

16. What is the logarithm of 9 in a system *whose base is* 3? Of 81? Of 27? Of 3? Of $\frac{1}{9}$? Of $\frac{1}{27}$?

17. What is the logarithm of 256 in a system *whose base is* 16? Of 16? Of 4? Of 8? Of 64?

18. What is the logarithm of 100 in a system *whose base is* 10? Of 1000? Of 100000? Of $\frac{1}{10}$? Of .01? Of 1? Of .001?

19. What is the logarithm of 81 in a system *whose base is* 27? Of 3? Of 9? Of 243? Of $\frac{1}{27}$? Of $\frac{1}{9}$? Of $\frac{1}{81}$?

259. The Briggsian or Common System of Logarithms uses 10 for its base.

1. Since $10^4 = 10{,}000$, then, $\log 10{,}000 = 4$ (§ 258).
$10^3 = 1{,}000$, then, $\log 1{,}000 = 3$.
$10^2 = 100$, then, $\log 100 = 2$.
$10^1 = 10$, then, $\log 10 = 1$.
$10^0 = 1$ (§ 185), $\log 1 = 0$.
$10^{-1} = .1$ (§ 185), $\log .1 = -1$.
$10^{-2} = .01$ $\log .01 = -2$.
$10^{-3} = .001$ $\log .001 = -3$.
etc. etc.

2. It is clear from the first two lines of 1 that any number between 1000 and 10,000 (as 6924.7), having four figures to the left of the decimal point, has for its logarithm $3 +$ a decimal, because its logarithm must lie *between* 3 and 4. Again, any number between 100 and 1000, having three figures to the left of the decimal point, has for its logarithm $2 +$ a decimal, because its logarithm must lie between 2 and 3; and any number between 10 and 100, having two integral orders, has for its logarithm $1 +$ a decimal, because its logarithm must lie between 1 and 2; and finally, any number between 1 and 10, as 5.698, having but one integral order, has for its logarithm $0 +$ a decimal, because its logarithm must lie between 0 and 1. Hence, generally

3. *The characteristic of the logarithm of any number greater than 1 is one less than the number of its integral orders.*

Thus, the characteristic of the logarithm of 729.4 is 2; of 7460 is 3; of 3.96 is 0.

4. In the common system, in which the base is 10, *the mantissas do not change when the decimal point is moved.*

To understand why this is so, take $10^{.1038} = 1.27$. Multiplying or dividing the members of this equation by $10^2 = 100$, or $10 = 10$, etc., we have, by the laws of exponents, §§ 42, 48:

$10^{2.1038} = 127$ (Mult. Ax.), or, $\log 127 = 2.1038$.

$10^{1.1038} = 12.7$ (Mult. Ax.), or, $\log 12.7 = 1.1038$.

$10^{.1038-1} = .127$ (Div. Ax.), or, $\log .127 = \bar{1}.1038$.

$10^{.1038-2} = .0127$ (Div. Ax.), or, $\log .0127 = \bar{2}.1038$.

etc. etc.

The minus signs *over* the characteristics at the right belong to the characteristics only. Thus, by regarding the characteristics only as changing signs, mantissas stay the same no matter where the decimal point in the number is changed to, and *mantissas are always positive.*

5. By examining the last two lines of the table in 1 of this article we see that any number between .001 and .01, having 2 ciphers before its first significant figures, has -3 for characteristic, since its logarithm lies between -3 and -2 and the mantissa added is positive. Again, any number between .01 and .1, having one cipher before its first significant figure, has -2 for its characteristic, since its logarithm lies between -2 and -1 and the mantissa added is positive; also, any number between .1 and 1, having *no* cipher before its first significant figure, has -1 for characteristic, since its logarithm lies between -1 and 0 and the mantissa added is positive. Hence, generally

6. *The characteristic of the logarithm of any number between* 1 *and* 0 *is negative and* 1 *more than the number of ciphers preceding its first significant figure.*

Thus, the characteristic of the logarithm of .00468 is -3; of .3794 is -1; of .000067 is -5; etc.

260. Explanation of the Four Place Logarithmic Table. The Table (see pp. 236, 237) gives the mantissa only of all the numbers from 100 to 999. The first two figures of the numbers are found in the column marked "N"; the third is one of the ten figures at the top of the table. Thus, the mantissa of 487 is found by taking the 48 in the "N" column and the 7 from the topmost row of figures; the mantissa sought is in the 7 column opposite 48 and is .6875. The characteristic of 487 is 2 (§ 259, 3), so that log 487 = 2.6875.

At the intersection of lines and columns are found the 900 mantissas corresponding to the 900 numbers from 100 to 999, the first two figures determining the line and the last the column. A decimal point before each mantissa is understood.

The columns of numbers marked "D" are merely the differences between the mantissas. Thus, the difference between .3032 and .3054 is 22 as in the table; and between .4133 (corresponding to 259) and .4150 (corresponding to 260) is 17.

This table gives directly the mantissas of all numbers consisting of three figures, preceded or followed by any number of ciphers. Thus, the mantissa of 31 is the same as that for 310, and is .4914 (see § 259, 4); the mantissa of the logarithm of 8 is the same as that for 800; the mantissa for 320000 or for .032 is the same as that for 320.

261. Exercise in finding the Logarithms of Numbers whose Mantissas are given directly in the Table. Common fractions and mixed numbers must be reduced to decimals.

Find the logarithms of:

1. 329. *Ans.* 2.5172. See § 259, 3 and § 260.

2. 764. 3. 125. 4. 969. 5. 370. 6. 37.

7. 400. 8. 40. 9. 4. 10. 7. 11. 70.

12. 700. 13. 7000. 14. .0372. 15. .000561. 16. .000029.

17. .002. 18. 3680. 19. $\frac{1}{4}$. 20. $\frac{1}{25}$. 21. $4\frac{1}{2}$.

22. $47\frac{1}{5}$. 23. $12\frac{1}{2}$. 24. $.06\frac{1}{2}$. 25. $.003\frac{3}{4}$. 26. $\frac{7}{8}$.

N	0	D	1	D	2	D	3	D	4	D	5	D	6	D	7	D	8	D	9	D
10	0000	43	0043	43	0086	42	0128	42	0170	42	0212	41	0253	41	0294	40	0334	40	0374	40
11	0414	39	0453	39	0492	39	0531	38	0569	38	0607	38	0645	37	0682	37	0719	36	0755	37
12	0792	36	0828	36	0864	35	0899	35	0934	35	0969	35	1004	34	1038	34	1072	34	1106	33
13	1139	34	1173	33	1206	33	1239	32	1271	32	1303	32	1335	32	1367	32	1399	31	1430	31
14	1461	31	1492	31	1523	30	1553	31	1584	30	1614	30	1644	29	1673	30	1703	29	1732	29
15	1761	29	1790	28	1818	29	1847	28	1875	28	1903	28	1931	28	1959	28	1987	27	2014	27
16	2041	27	2068	27	2095	27	2122	26	2148	27	2175	26	2201	26	2227	26	2253	26	2279	25
17	2304	26	2330	25	2355	25	2380	25	2405	25	2430	25	2455	25	2480	24	2504	25	2529	24
18	2553	24	2577	24	2601	24	2625	23	2648	24	2672	23	2695	23	2718	24	2742	23	2765	23
19	2788	22	2810	23	2833	23	2856	22	2878	22	2900	23	2923	22	2945	22	2967	22	2989	21
20	3010	22	3032	22	3054	21	3075	21	3096	22	3118	21	3139	21	3160	21	3181	20	3201	21
21	3222	21	3243	20	3263	21	3284	20	3304	20	3324	21	3345	20	3365	20	3385	19	3404	20
22	3424	20	3444	20	3464	19	3483	19	3502	20	3522	19	3541	19	3560	19	3579	19	3598	19
23	3617	19	3636	19	3655	19	3674	18	3692	19	3711	18	3729	18	3747	19	3766	18	3784	18
24	3802	18	3820	18	3838	18	3856	18	3874	18	3892	17	3909	18	3927	18	3945	17	3962	17
25	3979	18	3997	17	4014	17	4031	17	4048	17	4065	17	4082	17	4099	17	4116	17	4133	17
26	4150	16	4166	17	4183	17	4200	16	4216	16	4232	17	4249	16	4265	16	4281	17	4298	16
27	4314	16	4330	16	4346	16	4362	16	4378	15	4393	16	4409	16	4425	15	4440	16	4456	16
28	4472	15	4487	15	4502	16	4518	15	4533	15	4548	16	4564	15	4579	15	4594	15	4609	15
29	4624	15	4639	15	4654	15	4669	14	4683	15	4698	15	4713	15	4728	14	4742	15	4757	14
30	4771	15	4786	14	4800	14	4814	15	4829	14	4843	14	4857	14	4871	15	4886	14	4900	14
31	4914	14	4928	14	4942	13	4955	14	4969	14	4983	14	4997	14	5011	13	5024	14	5038	13
32	5051	14	5065	14	5079	13	5092	13	5105	14	5119	13	5132	13	5145	14	5159	13	5172	13
33	5185	13	5198	13	5211	13	5224	13	5237	13	5250	13	5263	13	5276	13	5289	13	5302	13
34	5315	13	5328	12	5340	13	5353	13	5366	12	5378	13	5391	12	5403	13	5416	12	5428	13
35	5441	12	5453	12	5465	13	5478	12	5490	12	5502	12	5514	13	5527	12	5539	12	5551	12
36	5563	12	5575	12	5587	12	5599	12	5611	12	5623	12	5635	12	5647	11	5658	12	5670	12
37	5682	12	5694	11	5705	12	5717	12	5729	11	5740	12	5752	11	5763	12	5775	11	5786	12
38	5798	11	5809	12	5821	11	5832	11	5843	12	5855	11	5866	11	5877	11	5888	11	5899	12
39	5911	11	5922	11	5933	11	5944	11	5955	11	5966	11	5977	11	5988	11	5999	11	6010	11
40	6021	10	6031	11	6042	11	6053	11	6064	11	6075	10	6085	11	6096	11	6107	10	6117	11
41	6128	10	6138	11	6149	11	6160	10	6170	10	6180	11	6191	10	6201	11	6212	10	6222	10
42	6232	11	6243	10	6253	10	6263	11	6274	10	6284	10	6294	10	6304	10	6314	11	6325	10
43	6335	10	6345	10	6355	10	6365	10	6375	10	6385	10	6395	10	6405	10	6415	10	6425	10
44	6435	9	6444	10	6454	10	6464	10	6474	10	6484	9	6493	10	6503	10	6513	9	6522	10
45	6532	10	6542	9	6551	10	6561	10	6571	9	6580	10	6590	9	6599	10	6609	9	6618	10
46	6628	9	6637	9	6646	10	6656	9	6665	10	6675	9	6684	9	6693	9	6702	10	6712	9
47	6721	9	6730	9	6739	10	6749	9	6758	9	6767	9	6776	9	6785	9	6794	9	6803	9
48	6812	9	6821	9	6830	9	6839	9	6848	9	6857	9	6866	9	6875	9	6884	9	6893	9
49	6902	9	6911	9	6920	8	6928	9	6937	9	6946	9	6955	9	6964	8	6972	9	6981	9
50	6990	8	6998	9	7007	9	7016	8	7024	9	7033	9	7042	8	7050	9	7059	8	7067	9
51	7076	8	7084	9	7093	8	7101	9	7110	8	7118	8	7126	9	7135	8	7143	9	7152	8
52	7160	8	7168	9	7177	8	7185	8	7193	9	7202	8	7210	8	7218	8	7226	9	7235	8
53	7243	8	7251	8	7259	8	7267	8	7275	9	7284	8	7292	8	7300	8	7308	8	7316	8
54	7324	8	7332	8	7340	8	7348	8	7356	8	7364	8	7372	8	7380	8	7388	8	7396	8

N	0	D	1	D	2	D	3	D	4	D	5	D	6	D	7	D	8	D	9	D
55	7404	8	7412	7	7419	8	7427	8	7435	8	7443	8	7451	8	7459	7	7466	8	7474	8
56	7482	8	7490	7	7497	8	7505	8	7513	7	7520	8	7528	8	7536	7	7543	8	7551	8
57	7559	7	7566	8	7574	8	7582	7	7589	8	7597	7	7604	8	7612	7	7619	8	7627	7
58	7634	8	7642	7	7649	8	7657	7	7664	8	7672	7	7679	7	7686	8	7694	7	7701	8
59	7709	7	7716	7	7723	8	7731	7	7738	7	7745	7	7752	8	7760	7	7767	7	7774	8
60	7782	7	7789	7	7796	7	7803	7	7810	8	7818	7	7825	7	7832	7	7839	7	7846	7
61	7853	7	7860	8	7868	7	7875	7	7882	7	7889	7	7896	7	7903	7	7910	7	7917	7
62	7924	7	7931	7	7938	7	7945	7	7952	7	7959	7	7966	7	7973	7	7980	7	7987	6
63	7993	7	8000	7	8007	7	8014	7	8021	7	8028	7	8035	6	8041	7	8048	7	8055	7
64	8062	7	8069	6	8075	7	8082	7	8089	7	8096	6	8102	7	8109	7	8116	6	8122	7
65	8129	7	8136	6	8142	7	8149	7	8156	6	8162	7	8169	7	8176	6	8182	7	8189	6
66	8195	7	8202	7	8209	6	8215	7	8222	6	8228	7	8235	6	8241	7	8248	6	8254	7
67	8261	6	8267	7	8274	6	8280	7	8287	6	8293	6	8299	7	8306	6	8312	7	8319	6
68	8325	6	8331	7	8338	6	8344	7	8351	6	8357	6	8363	7	8370	6	8376	6	8382	6
69	8388	7	8395	6	8401	6	8407	7	8414	6	8420	6	8426	6	8432	7	8439	6	8445	6
70	8451	6	8457	6	8463	7	8470	6	8476	6	8482	6	8488	6	8494	6	8500	6	8506	7
71	8513	6	8519	6	8525	6	8531	6	8537	6	8543	6	8549	6	8555	6	8561	6	8567	6
72	8573	6	8579	6	8585	6	8591	6	8597	6	8603	6	8609	6	8615	6	8621	6	8627	6
73	8633	6	8639	6	8645	6	8651	6	8657	6	8663	6	8669	6	8675	6	8681	5	8686	6
74	8692	6	8698	6	8704	6	8710	6	8716	6	8722	5	8727	6	8733	6	8739	6	8745	6
75	8751	5	8756	6	8762	6	8768	6	8774	5	8779	6	8785	6	8791	6	8797	5	8802	6
76	8808	6	8814	6	8820	5	8825	6	8831	6	8837	5	8842	6	8848	6	8854	5	8859	6
77	8865	6	8871	5	8876	6	8882	5	8887	6	8893	6	8899	5	8904	6	8910	5	8915	6
78	8921	6	8927	5	8932	6	8938	5	8943	6	8949	5	8954	6	8960	5	8965	6	8971	5
79	8976	6	8982	5	8987	6	8993	5	8998	6	9004	5	9009	6	9015	5	9020	5	9025	6
80	9031	5	9036	6	9042	5	9047	6	9053	5	9058	5	9063	6	9069	5	9074	5	9079	6
81	9085	5	9090	6	9096	5	9101	5	9106	6	9112	5	9117	5	9122	6	9128	5	9133	5
82	9138	5	9143	6	9149	5	9154	5	9159	6	9165	5	9170	5	9175	5	9180	6	9186	5
83	9191	5	9196	5	9201	5	9206	6	9212	5	9217	5	9222	5	9227	5	9232	6	9238	5
84	9243	5	9248	5	9253	5	9258	5	9263	6	9269	5	9274	5	9279	5	9284	5	9289	5
85	9294	5	9299	5	9304	5	9309	6	9315	5	9320	5	9325	5	9330	5	9335	5	9340	5
86	9345	5	9350	5	9355	5	9360	5	9365	5	9370	5	9375	5	9380	5	9385	5	9390	5
87	9395	5	9400	5	9405	5	9410	5	9415	5	9420	5	9425	5	9430	5	9435	5	9440	5
88	9445	5	9450	5	9455	5	9460	5	9465	4	9469	5	9474	5	9479	5	9484	5	9489	5
89	9494	5	9499	5	9504	5	9509	4	9513	5	9518	5	9523	5	9528	5	9533	5	9538	4
90	9542	5	9547	5	9552	5	9557	5	9562	4	9566	5	9571	5	9576	5	9581	5	9586	4
91	9590	5	9595	5	9600	5	9605	4	9609	5	9614	5	9619	5	9624	4	9628	5	9633	5
92	9638	5	9643	4	9647	5	9652	5	9657	4	9661	5	9666	5	9671	4	9675	5	9680	5
93	9685	4	9689	5	9694	5	9699	4	9703	5	9708	5	9713	4	9717	5	9722	5	9727	4
94	9731	5	9736	5	9741	4	9745	5	9750	4	9754	5	9759	4	9763	5	9768	5	9773	4
95	9777	5	9782	4	9786	5	9791	4	9795	5	9800	5	9805	4	9809	5	9814	4	9818	5
96	9823	4	9827	5	9832	4	9836	5	9841	4	9845	5	9850	4	9854	5	9859	4	9863	5
97	9868	4	9872	5	9877	4	9881	5	9886	4	9890	4	9894	5	9899	4	9903	5	9908	4
98	9912	5	9917	4	9921	5	9926	4	9930	4	9934	5	9939	4	9943	5	9948	4	9952	4
99	9956	5	9961	4	9965	4	9969	5	9974	4	9978	5	9983	4	9987	4	9991	5	9996	4

262. How to find Mantissas for Numbers not given in the Table.

1. Since changing the position of the decimal point in a number does not affect the value of the mantissa of its logarithm, to find the mantissas of numbers consisting of more than three significant figures, we may think of the decimal point as located so that we have a number consisting of three integral orders. Thus, if the given number is 46,792 we may think of it as changed to 467.92; if it is .006943, we may think of it as changed to 694.3.

2. Logarithms do not increase in proportion as their numbers increase.

Thus, $\log 3 = .4771$; $\log 6 = .7782$; $\log 12 = 1.0792$.

However, if we take large numbers close together, the increments of the numbers and the increments of their logarithms are very nearly proportional.

Thus, $\log 720 = 2.8573$; $\log 721 = 2.8579$; $\log 722 = 2.8585$; $\log 723 = 2.8591$.

Now, since log 721 is midway between log 720 and log 722 correct to 4 decimal places, we conclude that log 720.5 is probably about midway between log 720 and log 721, that is,

$$\log 720.5 = 2.8573 + .5 \text{ of } .0006 = 2.8576.$$

Reasoning in the same way, we infer that

$$\log 720.85 = 2.8573 + .85 \text{ of } .0006 = 2.8578.$$

Now, as the mantissas themselves are written as whole numbers and their differences are given as whole numbers, it is not necessary to write .0006, but merely 6, multiplying it by the decimal part of the given number. Then, taking the result to the nearest unit, we add it to the next lower mantissa in the table.

3. To find the mantissa of the logarithm of any number proceed as follows:

(1) *Find the mantissa corresponding to the first three figures of which the left one is significant, that is, is not* 0.

(2) *Multiply the tabular difference (column D, following the mantissa found) by the remaining figures of the number regarded as a decimal.*

(3) *Add the product, taken to the nearest unit, to the mantissa corresponding to the first three figures.*

This process is called **interpolation.**

263. Exercise in finding the Logarithms of Numbers.

1. Find the logarithm of 67.883.

SOLUTION. log 67.8 = 1.8312; .83 of 7 = 5.81 = 6 to the nearest unit; log 67.883 = 1.8318.

As soon as possible the whole operation ought to be performed mentally. Find the logarithms of the following:

2. 843.8. 3. 921.5. 4. 6.934. 5. 290.45. 6. 1764.

7. .1764. 8. .02596. 9. .20087. 10. .006784. 11. .00005.

12. 76349. 13. $11\frac{7}{8}$. 14. $\frac{5}{9}$. 15. $\frac{6}{170}$. 16. 3.1416.

17. 10007. 18. 12401. 19. 1.002. 20. $.1\frac{1}{8}$. 21. 2006.

264. How to find the Numbers corresponding to given Logarithms. Exercise. Numbers corresponding to given logarithms are often called the **antilogarithms** of the given logarithms.

1. Find antilog of 1.8142.

SOLUTION. The mantissa .8142 corresponds to 652. By the rule for characteristics, § 259, 8, there should be 2 integral orders in the answer. Hence, the answer is 65.2.

Find the antilogarithms of the following:

2. 2.6580 3. $\bar{2}.7388$ 4. 0.1959

5. Find the antilogarithm of 1.7963.

The process is the exact reverse of that of finding logarithms of numbers.

(1) *Look for that mantissa in the table which is next smaller than the given mantissa (in this case .7959, which corresponds to 625), and subtract this mantissa from the given one.*

(2) *Next divide this remainder by the tabular difference (7) following the mantissa found in the table, getting .57, and annex the quotient to the first three figures already found, getting 62557.*

(3) *Last of all locate the decimal point by the rules for characteristic (§ 259, 3, 6). Since the given characteristic is 1, there must be two integral orders in the answer,* i.e. *answer is 62.557.*

6. Find the antilogarithm of $\bar{3}.2432$.

Solution. $2432 - 2430 = 2$; $2 \div 25 = .08$. *Ans.* .0017508.

7. 2.8144. **8.** 5.7329. **9.** $\bar{2}.5394$. **10.** $\bar{6}.3139$.

11. 2.4774. **12.** $\bar{3}.6121$. **13.** 6.1920. **14.** $\bar{1}.0100$.

15. 0.0304. **16.** $\bar{2}.0021$. **17.** $\bar{6}.9779$. **18.** 0.9988.

19. $8.2463 - 10$. This form is often used instead of $\bar{2}.2463$.

20. $9.3891 - 10$. **21.** $7.4284 - 10$. **22.** $6.3239 - 10$.

II. USES OF LOGARITHMS

265. Theory of Logarithms. As logarithms are exponents, they are governed by exactly the same laws as exponents. (§ 179, § 187.)

1. Addition and subtraction *cannot be performed* by common logarithms. Thus, a^2 and a^3 cannot be added or subtracted by working with 2 and 3.

2. Multiplication is performed by adding logarithms. (§ 42.)

Example. To multiply 721 by 369.

Solution.
$$\begin{array}{r} \log 721 = 2.8579 \\ \log 369 = 2.5670 \\ \hline \log 266000 = 5.4249 \end{array}$$
Ans. (Antilog of 5.4249.)

Explanation.
$$\begin{array}{rl} 721 = & 10^{2.8579} \\ \times\ 369 = & \times\ 10^{2.5670} \\ \hline 721 \times 369 = & 10^{5.4249} \end{array}$$

Explanation. The logarithms of 721 and 369 are obtained from the table and added; then the antilogarithm of the sum is found.

By actual multiplication 721 × 369 = 266,049.

a. We see here that a Four Place Table gives only four figures of the answer correct. Occasionally it will give only three correct, but usually it will give four and often five right. Logarithmic tables are constructed for varying needs. For practical purposes Four and Five Place Tables are commonly employed. Results can be had as accurate as desired by using proper kind of tables. Tables have been constructed containing as many as 15 figures in their logarithms. The theory, of course, is the same for all tables.

3. *To multiply numbers together, get the logarithm of each factor, add the results, and find the antilogarithm of the sum.*

4. Multiply 763 by 298 and check by actual multiplication. Also 3.245 and 63.29; also 93.29, 29.76, and 16.48.

5. Division is performed by subtracting logarithms. (§ 187, 3.)

Example. Divide 37.69 by 2.463.

Solution.
$$\begin{array}{r}\log 37.69 = 1.5762\\ \log 2.463 = 0.3914\\ \hline \log 15.304^{-} = 1.1848\end{array}$$

Ans. (Antilog of 1.1848.)

Explanation.
$$\begin{array}{r}37.69 = 10^{1.5762}\\ 2.463 = 10^{.3914}\\ \hline 37.69 \div 2.463 = 10^{1.1848}\end{array}$$

By actual division the quotient is 15.302+.

6. Divide 19.65 by 2.843 and check by actual division; also 25,941 by 16,713; 34.62 by 7.329; 84.63 by .6792; .6721 by .00325.

7. Raising a number to a power is performed by multiplying its logarithm by the exponent of the power and finding the antilogarithm of the product. (§ 187, 4.)

Example. Raise 17.64 to the fourth power.

Solution.
$$\begin{array}{r}\log 17.64 = 1.2465\\ \log 17.64^{4} = 4.9860\\ \log 96825 = 4.9860\end{array}$$

Ans.

Explanation.
$$\begin{array}{r}17.64 = 10^{1.2465}\\ 17.64^{4} = 10^{4.9860}\\ 96825 = 10^{4.9860}\end{array}$$

By actual (contracted, § 176) multiplication $17.64^4 = 96{,}827^{-}$.

8. Raise 13.25 to 3d power and check by contracted multiplication; also 9.2 to 4th power; 3^9; $.17^4$; 16^{12}; $.06^5$.

9. Extracting a root of a number is performed by dividing its logarithm by the index of the root and finding the antilogarithm of quotient. (§ 187, 5.)

EXAMPLE. Calculate $(29.34)^{\frac{1}{5}}$, or $\sqrt[5]{29.34}$.

SOLUTION. $\log 29.34 = 1.4675$ EXPLANATION. $29.34 = 10^{1.4675}$

$\log 29.34^{\frac{1}{5}} = 0.2935$ $29.35^{\frac{1}{5}} = 10^{.2935}$

$\log 1.9655 = 0.2935$ $1.9655 = 10^{.2935}$

By actual contracted multiplication $1.9655^5 = 29.315$.

10. Extract the cube root of 1763 and check by the method of § 174; $\sqrt[4]{1.953}$ (check by § 172); $\sqrt[7]{19.84}$; $\sqrt[9]{85.34}$ (check by § 174).

11. Cases 7 and 9 just given can be included under one head.

EXAMPLE. $(29382)^{\frac{4}{5}}$.

SOLUTION. $\log 29382 = 4.4680$; $\log 29382^{\frac{4}{5}} = \frac{4}{5} \times 4.4680 = 3.5744 =$ antilog 3753. *Ans.* $\therefore (29382)^{\frac{4}{5}} = 3753$.

12. Calculate $41.6^{\frac{3}{4}}$; $1.908^{\frac{3}{5}}$; $3984^{\frac{2}{7}}$; $1501^{\frac{4}{9}}$; $20.6^{\frac{3}{4}}$.

266. Exercise in Multiplication and Division by Logarithms. Check by contracted multiplication and division.

1. 26.73×19.38. 2. $26.45 \times .02687 \times 3.194$. 3. $.00857 \times .00693$.

4. $\dfrac{862 \times 48.75}{7.862 \times 6.827}$

SUGGESTION. Get logarithms of factors in numerator and add; also logarithms of factors in denominator and add; subtract latter sum from the former and get antilogarithm of difference.

5. $\dfrac{89.76 \times 98.54 \times 26.63}{.005862 \times 8271}$ 6. $\dfrac{87.51 \times 445 \times 823}{3.1416 \times .045 \times .862}$.

a. Carrying and borrowing where negative characteristics enter are likely to prove confusing to the student. It is generally simpler to add 10 to the characteristic and subtract 10 from the mantissa. Explain the following:

$$(1)\ \begin{array}{r} \bar{4}.3421 \\ +\,2.9745 \\ \hline \bar{1}.3166 \end{array} \qquad (2)\ \begin{array}{r} 2.6789 \\ -\,\bar{4}.8541 \\ \hline 5.8248 \end{array} \qquad (3)\ \begin{array}{r} \bar{3}.2681 \\ -\,\bar{2}.7492 \\ \hline \bar{2}.5189 \end{array}$$

Solve these exercises now by adding and subtracting 10 for each logarithm with a negative characteristic.

267. Powers and Roots by Logarithms. The case of negative exponents needs attention.

1. Raise .0734 to the fourth power. Check by § 176.

SOLUTION. $\log .0734^4 = (8.8657 - 10) \times 4 = 35.4628 - 40 = \bar{5}.4628.$

$\therefore .0734^4 = .000029.$

2. $.0634^5$. 3. $.637^4$. 4. $.007234^7$. 5. $.07^6$.

6. Extract the fifth root of .0329.

$$\log .0329 = 8.5172 - 10 = 48.5172 - 50.$$
$$5\,|\,\underline{48.5172 - 50}$$
$$\text{Antilog of } 9.7034 - 10 = .5051. \quad \textit{Ans.}$$

Notice that − 50 was chosen so as to be exactly divisible by 5.

7. $.0072^{\frac{1}{4}}$. 8. $.02852^{\frac{1}{3}}$. 9. $\sqrt{.6782}$. 10. $\sqrt[4]{.8751}$.

268. Cologarithms. The remainder obtained by subtracting a logarithm from 10 is called the **cologarithm** of the number. By means of cologarithms combined multiplications and divisions can be changed into multiplications. Thus, $a \times b \div c$ can be written $a \times b \times \frac{1}{c}$. Now $\log \frac{1}{c} = \log 1 - \log c = 0 - \log c$. Instead of subtracting $\log c$ from 0, it is more convenient to subtract it from 10; then 10 will have to be taken from the final result for each cologarithm introduced. The preceding may be stated as follows:

$$\log \frac{ab}{c} = \log a + \log b + (10 - \log c) - 10.$$

1. Calculate $\dfrac{36.74 \times 78.91}{19.83 \times 21.34}$, using cologarithms.

SOLUTION.

$$\begin{array}{rcc}
\log 36.74 = 1.5651 & \text{colog } 19.83 & \text{colog } 21.34 \\
\log 78.91 = 1.8972 & 10.0000 & 10.0000 \\
\text{colog } 19.83 = 8.7026 & \underline{1.2974} & \underline{1.3292} \\
\text{colog } 21.34 = \underline{8.6708} & 8.7026 & 8.6708 \\
\log 6.85 \;\; = 0.8357 & \multicolumn{2}{l}{\therefore 6.85 \text{ is answer sought.}}
\end{array}$$

2. Calculate $\frac{27.43 \times 168.4}{1.938 \times 247.6}$. 3. Calculate $\frac{176.4 \times 18.25}{3.1416 \times 14.3^2}$.

269. Exercise in applying Logarithms in the Solution of Practical Problems.

1. Find the circumference of a circle whose diameter is 17.63 inches, from the formula $c = \pi d$.

SUGGESTION. The logarithm of π is .4971. The student should *remember* this, as it will save him the trouble of looking it up each time.

2. Find the area of a circle whose radius is 16.72 ft., from the formula $a = \pi r^2$.

3. Find the diameter of a circle whose circumference is 3928 m.

4. Find the surface of a sphere whose radius is 362.5 in., from the formula $s = 4\pi r^2$.

5. Find the volume of a sphere whose diameter is 12 cm., from the formula $v = \frac{1}{6}\pi d^3$.

6. Find the radius of a sphere whose surface is 25.12 sq. ft.

7. Find the area of an ellipse whose longer semiaxis a is 22.18 in. and shorter b is 16.88 in., from the formula $A = \pi ab$.

8. Find the volume of a cylinder whose radius of base r is 1.677 m. and whose length l is 23.51 m., from the formula $v = \pi r^2 l$.

9. Find the volume of a cone whose radius of base r is 33.41 ft. and altitude a is 12.16 ft., from the formula $v = \frac{1}{3}\pi r^2 a$.

10. Find area of a triangle whose sides are 16.35, 18.97, and 24.77, using formula p. 65.

11. Perform the multiplications and divisions called for in § 251 with logarithms.

The problems of §§ 251–255 can be solved by logarithms. For this purpose special tables are constructed which contain the logarithms of the trigonometical functions for every degree and minute from 0° to 90°. Thus, this table contains the logarithms of the numbers in the table on page 224. By the use of such a table with an ordinary table of logarithms, problems in right triangles can be solved very quickly.

270. Compound Interest and its Calculation by Logarithms. — In compound interest the amount becomes the principal at the beginning of each new interest period. Call p the principal, a the final amount, r the rate, and n the number of years.

1. *To find the amount when the principal, rate, and time are given.* The amount at the end of one year is $p + pr$, or $p(1+r)$, since p is the principal and pr is one year's interest. Thus, to get the amount at the end of one year always multiply the principal by $1 +$ the rate.

Now, in compound interest the principal at the beginning of the second year is $p(1+r)$. Then the amount at the end of the second year is $p(1+r)(1+r)$, or $p(1+r)^2$; and so on for the n years. Hence

$$a = p(1+r)^n.$$

Example. Find the amount of \$725.15 at 5% compound interest at the end of 6 years.

Solution. $a = 725.15(1+.05)^6$.

$$\begin{array}{rl}
\log 1.05 &= .0212 \\ \hline
\log 1.05^6 &= .1272 \\
\log 725.15 &= 2.8604 \\ \hline
\log a &= 2.9876 \\
a &= 971.80. \quad \textit{Ans.}
\end{array}$$

a. A six- or seven-place table is needed if the answer is to be obtained correct to cents.

2. *To find the cost or "present worth" of a sum payable n years hence, supposing interest to be compounded.*

From the equation just obtained by solving for p, we get

$$p = \frac{a}{(1+r)^n}.$$

Example. Find the principal which will amount to \$923 at 4% compound interest in 12 years, by substituting in the formula.

3. *To find amount when interest is compounded q times a year.* Explain derivation of formula that follows.

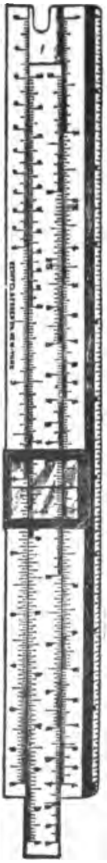
$$a = p\left(1 + \frac{r}{q}\right)^{qn}.$$

271. Exercise in solving Compound Interest Problems.

1. Find amount of $ 933 at 5 % which ran 7 yr.
2. Find amount of $ 336.1 at 6 % which ran 14 yr.
3. Find amount of $ 182.70 at 4½ % running 11 yr.
4. Find amount of $ 2673 at 3 % running 22 yr.
5. Find amount of $ 193.60 at 6 % running 8 yr.
6. Find principal which amounts to $ 775.20 in 15 yr. at 5 %.
7. Find principal which amounts to $ 9675 in 20 yr. at 4 %.
8. Find cost or present worth of $ 918 to be paid in 10 yr., allowing 5 % interest.
9. Find the amount of $ 225 at 5 % running 75 yr.
10. Find the amount of $ 700 which ran for 12 years, the interest being compounded semiannually at 6 %.

SUGGESTION. $a = p\left(1 + \frac{r}{2}\right)^{2n}$. See § 271, 3.

11. Find amount of $ 425 compounded quarterly at 7 %, which ran 9 years.
12. Find what is due in 1911 on a note calling for 1¢ as principal made in the time of Julius Cæsar 50 years before Christ, if 4 % compound interest is allowed.



272. Slide Rule. Rules are in common use which have for the figures given on them the left column of the logarithmic table, but instead of the mantissas being denoted by figures, these mantissas are given by their *lengths*. In using these rules one is virtually using a table of logarithms. See drawing in the margin.

III. EXPONENTIAL EQUATIONS

273. Solution of Exponential Equations. An **exponential equation** is one in which the unknown quantity is an exponent. Thus, $20^x = 75$ is an exponential equation. Such equations are solved by means of logarithms.

Let $a^x = b$ represent a simple exponential equation.

Then, $\log a^x = \log b$, (Logarithms of equals are equal.)

or, $x \times \log a = \log b$. (The logarithm of a power equals logarithm of the quantity multiplied by the exponent of the power, § 265, 7.)

$$\therefore x = \frac{\log b}{\log a}. \quad \text{(Div. Ax.)}$$

1. Solve $8^x = 45$ for x.

SOLUTION. $x = \dfrac{\log 45}{\log 8} = \dfrac{1.6532}{.9031} = 1.83+$

Solve the following equations:

2. $3^x = 29$.
3. $10^x = 129$.
4. $17^x = 1174$.
5. $123^x = 1684$.
6. $2^x = 25$.
7. $100^x = 8$.

a. The formula calls for getting the logarithms of a and b out of the table and *dividing* the latter by the former. This division can be performed by logarithms, but it is better to execute this division by the old familiar process of long division at first, and until the formula is well fixed in mind. The student may now test his divisions in the preceding problems by a logarithmic solution, and hereafter perform the divisions by logarithms.

8. $15.49^x = 6$.
9. $127^y = 4675$.
10. $3^x = 729$.

b. Exponential equations can have any positive value except unity for a and b. We have avoided the use of negative characteristics in the preceding problems, since they must be handled with care.

11. $12^x = .5$. SOLUTION. $x = \dfrac{9.6990 - 10}{1.0792} = \dfrac{-.3010}{1.0792} = -.28^-$.
12. $15^x = .42$.
13. $.9^x = 25$.
14. $.85^x = .687$.
15. $10^{2x} + 2 \times 10^x = 80$.
16. $2^x + \dfrac{1}{2^x} = 5$.

274. Historical Notes. Logarithms were invented by John Napier, of Merchiston, Scotland, about 1614 A.D. Napier's logarithms were not logarithms of ordinary numbers but of the ratios of the legs of a right-angled triangle to the hypotenuse, and decreased as these numbers increased. Napier's logarithms are very curious indeed. To understand why they were thought of rather than those we have, it must be kept in mind *that exponents were not known or used in those days.*

Henry Briggs (later of Oxford University, England) greatly admired Napier's logarithms and was led to visit him. The scene at their meeting was impressive. When Briggs came into Napier's presence, they greeted each other, and then almost a quarter of an hour elapsed with each looking at the other and not speaking a word. Then Briggs said, "My lord, I have undertaken this long journey purposely to see your person and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy; but, my lord, being by you found out, I wonder nobody found it out before, when now known it is so easy."

Later, Briggs constructed tables of logarithms of numbers, as we have them, to base 10. His table was calculated to 14 decimal places, and extended from 1 to 20,000 and from 90,000 to 100,000. The gap in Briggs's table was filled later (1628) by Adrian Vlacq, of Gouda, Holland, who calculated the list of missing logarithms.

The advantage in the use of logarithms in calculations was soon recognized, and by 1630 they were in general use in Europe.

In the history of English science Napier's book comes next in importance after Newton's great work on mechanics, the "Principia." Napier's invention was not the result of a happy accident but of long continued endeavor, and it is the more remarkable because made in what was a troubled age, in a rather wild country judged by our standards, and at a time when all science was in a most crude state.

For fuller accounts of development of logarithms see "Logarithms," "Napier," "Briggs," etc., in *Encyclopædia Britannica.*

JOHN NAPIER (1550-1617)

CHAPTER XI

ARITHMETICAL AND GEOMETRICAL PROGRESSIONS

I. ARITHMETICAL PROGRESSION

275. An Arithmetical Progression is a series of terms that increase or decrease by a common difference.

E.g. in the series 4, 7, 10, 13, 16, etc., the common difference is 3.

276. Formula for finding the *n*th Term of an Arithmetical Series. Let a be the first term, d the common difference, n the number of terms, and l the last or nth term. Then, by definition, the series is

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,\ l-2d,\ l-d,\ l,$$

where the dots, as usual, signify terms left out. To find the nth term, let us write the number of each term over it. Thus,

$$\overset{1}{a},\ \overset{2}{a+d},\ \overset{3}{a+2d},\ \overset{4}{a+3d},\ \overset{5}{a+4d},\ \cdots,\ \overset{n}{a+(n-1)d}.$$

The last term is found by observing that the coefficient of d is always 1 less than the number of the term over it. Hence,

$$(1)\quad l=a+(n-1)d.$$

277. Formula for Finding the Sum (s) of all the terms in an Arithmetical Progression. The sum can be found by writing the series *reversed* under it. Thus,

$$\begin{aligned} s&=a+a+d+a+2d+a+3d+\cdots+l-d+l\\ s&=l+l-d+l-2d+l-3d+\cdots+a+d+a\\ \hline 2s&=a+l+a+l+a+l+a+l+\cdots+l+a+l+a. \quad \text{(Add. Ax.)}\end{aligned}$$

$\therefore 2s = n(a+l)$, since there are evidently n of the $(a+l$'s) added. Then,

$$(2)\quad s = \frac{n}{2}(a+l).$$

Make *theorems* out of formulas (1) and (2).

278. Exercise in Arithmetical Progression.

1. Find the tenth term of 2, 11, 20, ⋯

SOLUTION. In this problem $a = 2$, $d = 9$, $n = 10$. Then, using equation (1),

$$l = 2 + (10 - 1)9 = 83.$$

This answer can be checked by writing the series out. Thus, 2, 11, 20, 29, 38, 47, 56, 65, 74, 83; 83 is the 10th term.

2. Find the 12th term of 4, 6, 8, ⋯.

3. Find the 15th term of 144, 138, 132, ⋯.

SUGGESTION. Put $d = -6$.

4. Find the 25th term of $\frac{15}{8}$, $\frac{21}{8}$, $\frac{27}{8}$, ⋯.

5. Find the 17th term of .5, .3, .1, ⋯

6. Find the sum of 11 terms of 18, 21, 24, ⋯

SUGGESTION. First find l as in the preceding. Then, using equation (2), find s.

7. Find the last term and sum in 5, 9, 13, etc., to 19 terms.

8. Simplify $7 + \frac{29}{4} + \frac{15}{2} +$ etc., to 16 terms.

9. Sum $2a - 5b$, $7a - 2b$, etc., to 9 terms.

10. Find the sum of 20 terms of 5, 1, -3, etc.

11. Given $a = 94$, $n = 12$, $d = -3$, to find l and s.

12. Insert 12 terms *between* 12 and 77.

SUGGESTION. Here $a = 12$, $l = 77$, $n = 14$; to find d.

13. The first term of a series is 2 and the common difference is $\frac{1}{8}$. What is the number of the term that is 10?

14. Find the sum of $-3q$, $-q$, q, etc., to p terms.

SUGGESTION. $n = p$, $d = 2q$.

15. Given $n = 7$, $d = 15$, $s = 399$, to find a and l.

Suggestion. Substitute these values in *both* equations (1) and (2). Then eliminate either a or l between the resulting equations.

16. Given $a = 10$, $d = 7$, $s = 582$, to find l and n.

Suggestion. After substituting as before and eliminating l, a quadratic equation, with n as its unknown, results.

17. Given l, a, d, to find n. Suggestion. Solve (1) for n.

18. Given l, a, n, to find d; given l, n, d, to find a; given s, n, l, to find a; given s, n, a, to find l; given s, l, a, to find n.

19. Given d, n, s, to find l.

Suggestion. Eliminate the missing letter a between (1) and (2), and solve the resulting equation for l.

20. Given a, d, n, to find s; given a, d, l, to find s; given d, n, l, to find s; given a, n, s, to find d; given a, l, s, to find d; given n, l, s, to find d; given d, n, s, to find a.

21. Given a, d, s, to find l.

Solution. These four letters not appearing by themselves in either (1) or (2), we eliminate the *missing letter* n from them.

$$(2_1) \quad n = \frac{2s}{a+l}.$$

$$(1) \quad l = a + \left(\frac{2s}{a+l} - 1\right)d. \qquad (\S\ 143.)$$

$$l = a + \frac{2ds - ad - ld}{a+l}.$$

$$al + l^2 = a^2 + al + 2ds - ad - ld. \qquad (\text{Ax. ?})$$

$$l^2 + dl = a^2 - ad + 2ds. \qquad (\text{Ax. ?})$$

$$l^2 + dl + \frac{d^2}{4} = a^2 - ad + \frac{d^2}{4} + 2ds. \qquad (\text{Ax., } \S\ 210.)$$

$$l + \frac{d}{2} = \sqrt{\left(a - \frac{d}{2}\right)^2 + 2ds}. \qquad (\text{Ax. ?})$$

$$l = -\frac{d}{2} \pm \sqrt{\left(a - \frac{d}{2}\right)^2 + 2ds}. \qquad (\text{Ax. ?})$$

22. Given $a = -7$, $d = 3$, $s = 430$, to find l by substituting in the formula just found.

23. Given l, d, s, to find a; given d, l, s, to find n; given d, a, s, to find n.

24. In a potato race 100 potatoes are placed 2 ft. apart in a straight line. A runner starting from the basket picks up one potato at a time and carries it to a basket in a line with the potatoes and 10 ft. from the first potato. How far must the contestant run?

25. How many strokes does the hammer on the bell of a clock make in a 24-hour day?

26. A body falling freely falls 16.08 ft. the first second, and in each succeeding second 32.16 ft. more than in the second immediately preceding. If a stone dropped from a stationary balloon reaches the ground in 14 sec., how far does it fall the last second and how high is the balloon?

27. A stone is dropped from the top of a tower 402 ft. high. In how many seconds does it reach the ground?

28. If a bullet when fired vertically upward goes 1008 ft. the first second, how high does it rise, and how long will it be till it reaches the earth again? In this problem use 16 and 32 instead of 16.08 and 32.16.

29. A farmer agreed to pay a blacksmith for shoeing his horse $\frac{1}{4}$¢ for the first nail he drove, $\frac{1}{2}$¢ for the next, $\frac{3}{4}$¢ for the third, and so on. There were 40 nails. How much did he have to pay?

30. If a person saves $100 and puts it at simple interest at 5% at the end of each year, how much will his property amount to at the end of 20 years?

31. A man was paid for drilling an artificial well 3.24 marks for the first meter, 3.29 marks for the second, 3.34 marks for the third, and so on. The well had to be sunk 500 meters. How much was paid for the last meter, and how much for the whole?

32. A travels uniformly 20 mi. a day; B starts 3 da. later and travels 8 mi. the first day, 12 mi. the second, and so on, in arithmetical progression. In how many days will B overtake A ?

SUGGESTION. Let x = number of days B travels to overtake A.
Then how many days does A travel ? What is his whole distance ?
With $n = x$, sum the progression and then make the equation and solve it.

33. The sum of three numbers in arithmetical progression is 12, and the sum of their squares is 66. Find them.

II. GEOMETRICAL PROGRESSION

279. **A Geometrical Progression** is a series in which the terms increase or decrease by a common *ratio.*

E.g. 3, 6, 12, 24, ⋯, the common ratio being 2.

280. Formula for finding the Last Term of a Geometrical Progression.

Let a be the first term, r the ratio, n the number of terms, l the last term, and s the sum of all the terms. Then, by definition, the series is,

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & & n \\ a, & ar, & ar^2, & ar^3, & ar^4, & ar^5, & \cdots, & ar^{n-1} \end{array}$$

The last term, ar^{n-1}, is found by noticing that the exponent of r is always 1 less than the number of the term (written over it in the series given above).

Hence, $$(1)\quad \boldsymbol{l = ar^{n-1}}.$$

281. Formula for finding the Sum of a Geometrical Progression.

To derive this sum, we write s = the sum of the series, and on the next line, the first equation multiplied by r.

$$\begin{array}{lll} s = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-2} + ar^{n-1} & & \\ rs = \quad\; ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-2} + ar^{n-1} + ar^n & & \text{(Ax. ?)} \\ \hline rs - s = ar^n - a. & & \text{(Sub. Ax. ?)} \\ \therefore (r-1)s = ar^n - a. & & \text{(On factoring.)} \\ (2)\; s = \dfrac{a(r^n - 1)}{r-1}. & & \text{(Ax. ?)} \end{array}$$

Explanation. As already stated, the equation over the long line, p. 254, is obtained by multiplying the first equation by r. Thus, r times the left member gives rs. Now, when we get the product of the right side of the first equation multiplied by r, we write the product of a and r underneath the ar of the first line, and the product of ar and r underneath ar^2 of the first line, and so on. The ar^{n-2} term of the second line comes from a missing term (indicated by dots) in the first line. When we subtract the first equation from the second, all the terms cancel except the first and last.

Again, since $l = ar^{n-1}$, $lr = ar^{n-1} \times r = ar^n$. Hence, substituting lr for ar^n in equation (2), p. 254, we have

$$(3)\quad s = \frac{lr - a}{r - 1}.$$

Make theorems out of formulas (1), (2), and (3).

282. Summation of a Geometrical Progression the Limit of whose Last Term is 0. (The last term is really an *infinitesimal*, or number smaller than any that can be named.) Using eq. (3), § 281, we have,

$$(4)\quad \text{Limit of } s = \frac{0 \times r - a}{r - 1} = \frac{-a}{r - 1} = \frac{a}{1 - r}.$$

1. Sum the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$ to infinity.

Solution. — Here $a = 1$, $r = \frac{1}{2}$ (2d term ÷ 1st term), and we put $l = 0$.

$$s = \frac{1}{1 - \frac{1}{2}} = 2. \quad \textit{Ans.}$$

2. Sum the series $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}$, to infinity.

3. Sum the repeating decimal .282828…, that is find the common fraction which reduced to a decimal gives these figures.

Solution. $a = .28$, $r = .01$, $l = 0$. Then,

$$s = \frac{.28}{1 - .01} = \frac{.28}{.99} = \frac{28}{99}. \quad \textit{Ans.}$$

Proof: $28 \div 99 = .282828\cdots$.

4. Sum the repeating decimal .43782782….

Suggestion. Regard .00782 as the first term, and add the answer to .43. What does r equal here?

283. Exercise in Geometrical Progression.

1. Find the 8th term of 6, 12, 24, ⋯.

2. Find the 7th term of 7, 21, 63, ⋯.

3. Find the 9th term and sum of nine terms of 1, 3, 9, ⋯.

4. Find the 10th and 16th terms of 256, 128, 64, ⋯.

5. Find the last term and sum of 8.1, 2.7, .9, ⋯ to 7 terms.

6. Find the last term and sum of $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, \cdots$ to 12 terms.

7. Find the last term and sum of $-\frac{2}{5}, \frac{1}{2}, -\frac{5}{8}, \cdots$ to 6 terms.

8. Sum to n terms $5, 2\frac{1}{2}, 1\frac{1}{4}, \cdots$. Suggestion. Here $n = n$.

9. Sum to infinity $3 + \frac{3}{2} + \frac{3}{4} + \cdots$.

10. Sum to infinity $4 + 2\frac{2}{3} + 1\frac{7}{9} + \cdots$.

11. Find the common fraction value of .3333⋯.

12. Reduce to a common fraction .3787878⋯.

13. Given $a = 8$, $r = 2$, $s = 248$, find l and n.

14. Given $r = 6$, $n = 5$, $l = 1296$, find a and s.

15. Given $r = 2$, $n = 12$, $l = 43008$, find a and s.

16. Insert 6 geometrical means between 56 and $-\frac{7}{16}$.

17. The 4th term of a geometrical progression is 160 and the ratio is 4; find the 7th and 15th terms.

18. A man deposited $500 in a savings bank that paid 4% interest, compounded annually. If the money was left 4 years, what sum stood to the depositor's credit?

Suggestion. $r = 1.04$, $n = 5$. The $500 is *first* term.

19. In using an air pump the pressure is reduced $\frac{1}{8}$ at each stroke and stands at 30 in. at the beginning. What is the height of the mercury after the fifth stroke?

Suggestion. $r = \frac{7}{8}$, $n = 6$.

20. If a rubber ball rebounds $\frac{1}{4}$ of the distance it falls each time, what will be the distance a ball moves that is thrown up with a force to carry it 75 ft.? (See § 282.)

21. If a man, whether by his example or designedly, leads a single fellow-man into the path of rectitude each year during 20 years, and each of these men in turn leads one man aright each year from the time of his change, and so for every one affected, what will be the total number influenced as a result of the first man's effort at the end of the twenty years?

22. Achilles pursued a tortoise, which was one stadium ahead of him, with a speed 12 times greater than that of the tortoise. When Achilles reached the place where the tortoise had been when he started, the tortoise was still $\frac{1}{12}$ stadium in advance of him; Achilles having traversed this distance, the tortoise was still $\frac{1}{144}$ stadium ahead of him, and so on, *ad infinitum*. Will Achilles then never catch up with the tortoise?

NOTE. This problem is a statement of Zeno's celebrated sophism. If x = number of stadia Achilles must run to catch tortoise, show that $x = 1\frac{1}{11}$. While it is perfectly allowable *to conceive* of the distance divided up in the way described, the sum of the infinite number of parts is only $1\frac{1}{11}$ stadia. Achilles will catch the tortoise in the time it takes him to run $1\frac{1}{11}$ stadia. The sophism consists in implying a finite period of time is needed for each division of $(1 + \frac{1}{12} + \frac{1}{144}$, etc.) stadia.

23. Seven old women go to Rome; each woman has 7 mules, each mule carries 7 sacks, each sack contains 7 loaves, with each loaf are 7 knives, each knife is put in 7 sheaths. What is the sum total of all? (From Leonardo's *Liber Abaci.*)

24. The population of a city is 100,000, and it increases 50 % every 4 years. What will the population be in 20 years?

25. A person has two parents, each of his parents has two parents, and so on. How many ancestors has a person, going back ten generations, counting his great-grandparents as the first generation (and assuming that each ancestor is an ancestor in only one line of descent)?

NOTE. — Before leaving the subject the pupil should be tested to see if he can distinguish between arithmetical and geometrical progression problems.

CHAPTER XII

BINOMIAL THEOREM

I. MATHEMATICAL INDUCTION

284. Mathematical Induction. — In the proof of the binomial theorem, which follows, use is made of **mathematical induction.** Mathematical induction, like induction in natural science, proceeds from the particular to the general, but it differs from induction in natural science in that the generalization is complete and absolute in its application, while in natural science the induction is very rarely, or never, complete. To make the nature of the method clear let us consider some examples of its application.

1. Show by mathematical induction that the *sum* of n terms of the series

$$\frac{1}{1\times 2},\ \frac{1}{2\times 3},\ \frac{1}{3\times 4}\cdots \text{is } \frac{n}{n+1}$$

PROOF. Let $S_1 \equiv \frac{1}{1\times 2} = \frac{1}{2}$; $S_2 \equiv \frac{1}{1\times 2} + \frac{1}{2\times 3} = \frac{2}{3}$; etc.

We see that *the value of each S* (sum) *examined has for its numerator the same as the S's subscript and for its denominator 1 more than its numerator.* Then, by this rule we should have

$$S_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n\times(n+1)} = \frac{n}{n+1}.$$

Now "S_{n+1}" means the sum of all the terms from the first to the $n+1$ *th.* This sum can be found by adding the next term to the value of S_n.

Thus, $$S_{n+1} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2} \qquad \text{(By addition.)}$$

Hence we see that *IF* $S_n = \frac{n}{n+1}$, it *follows* that $S_{n+1} = \frac{n+1}{n+2}$.

But we *know* $S_2 = \frac{2}{3}$, whence it follows that $S_3 = \frac{3}{4}$, and so on indefinitely.

Thus, the formula $S_n = \frac{n}{n+1}$ holds universally for this series, which was to be proved.

Notice particularly the argument on the last line of p. 258 and top of this page.

This method can be used for proving many such exercises. Thus, by it we can prove that

$$1^3 + 2^3 + 3^3 + \cdots \text{ to } n^3 = \left(\frac{n}{2}\right)^2 (n+1)^2.$$

2. Show by mathematical induction that $x^n - y^n$ is always exactly divisible by $x - y$.

PROOF.
$$\begin{array}{lll} & x^n - y^n & \underline{|x - y} \\ & \underline{x^n - x^{n-1}y} & \underline{|x^{n-1}} \\ & x^{n-1}y - y^n & \\ \text{or} & y(x^{n-1} - y^{n-1}) & \end{array}$$

Now, evidently if $x^{n-1} - y^{n-1}$ contains $x - y$, the remainder after the first division will, and so the whole division will come out without remainder.

Thus, we learn that if the difference of two same powers of x and y, viz. $x^{n-1} - y^{n\ 1}$ contains $x - y$, then the difference of the next higher powers, $x^n - y^n$, will exactly contain $x - y$.

But we know that $x^2 - y^2$ contains $x - y$; hence $x^3 - y^3$ contains $x - y$; then $x^4 - y^4$ contains $x - y$ exactly; and so on indefinitely. This theorem was also proved by means of the divisibility theorem on page 47, foot note.

3. Show by mathematical induction that $x^{2n} - y^{2n}$ contains $x + y$. (§ 71. See also Ex. 16, page 102.)

SUGGESTION. In this case the division will have to be carried on until two terms are found in the quotient.

4. Show by mathematical induction that $x^{2n+1} + y^{2n+1}$ contains $x + y$. (§ 72.)

SUGGESTION. Make this demonstration dependent on that of Ex. 3, getting only one term in the quotient.

a. The demonstrations by mathematical induction, as already stated, give results as universally true as by any other method. In physical science results obtained by induction always remain open to revision. Thus, while it would be found that water boils at 212° at a thousand places, we should find the rule breaking down at mountain heights. In mathematical induction all the particulars are arranged in a series like ninepins. We upset one of the nine pins and all the series goes down.

II. THE BINOMIAL THEOREM

285. The Binomial Coefficients. We saw in § 166 that the signs and exponents in raising binomials to powers gave little trouble, but that the *coefficients* were found with more difficulty.

Let us write down these coefficients for several powers. They are

for $(a+b)^1$,	1, 1
for $(a+b)^2$,	1, 2, 1
for $(a+b)^3$,	1, 3, 3, 1
for $(a+b)^4$,	1, 4, 6, 4, 1
for $(a+b)^5$,	1, 5, 10, 10, 5, 1
for $(a+b)^6$,	1, 6, 15, 20, 15, 6, 1

Pascal, an eminent French mathematician (1623–1662), observed that the coefficients in any line can be obtained from the line above it by adding the coefficient immediately above the required one to the one which precedes it.

Thus, 6 in the fourth power comes from adding 3 above it to the 3 just before this 3. The last 15 in the sixth power comes from adding 5 above it to the 10 which precedes the 5; and so on. This writing and deriving binomial coefficients is called "Pascal's triangle."

Sir Isaac Newton (1642–1727) conceived the idea of making a *formula* which would give directly the expansion of $(a+b)$ to any power, or any desired term in it, without the use of Pascal's triangle.

286. Development of Newton's Theorem for any Positive Integral Exponent.

Let us take n to denote the exponent of the desired power and use the rules of § 166 to write down the result in the same way in which we wrote down particular powers, such as the sixth, tenth, etc. In this way we get

$$(a-b)^n = a^n - \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\times 2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}b^3 + \text{etc.}$$

Thus, to get any coefficient we multiply the coefficient of the preceding term by the exponent of the leading letter, and divide by 1 more than the exponent of the other letter.

Now, while we saw that this rule held as far as we tried it, we do not *know* yet that it holds always, and we seek to prove that it is universally true. Instead of passing from the fourth to the fifth power, or from the fifth to the sixth, let us now try to pass from the nth to the $(n+1)$th power. To do this we multiply both sides of the equation just written by $a-b$. Then, for the left side of the new equation we have $(a-b)^n \times (a-b) = (a-b)^{n+1}$. For the right side, we have

$$
\begin{array}{lllll}
a^n & -\frac{n}{1}a^{n-1}b & +\frac{n(n-1)}{1\times 2}a^{n-2}b^2 & -\frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}b^3 & +\text{ etc.} \\
a & -b & & & \\
\hline
a^{n+1} & -\frac{n}{1}a^n b & +\frac{n(n-1)}{1\times 2}a^{n-1}b^2 & -\frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-2}b^3 & +\text{ etc.} \\
 & -a^n b & +\frac{n}{1}a^{n-1}b^2 & -\frac{n(n-1)}{1\times 2}a^{n-2}b^3 & +\text{ etc.} \\
\hline
a^{n+1} & -\frac{(n+1)}{1}a^n b & +\frac{(n+1)n}{1\times 2}a^{n-1}b^2 & -\frac{(n+1)(n)(n-1)}{1\times 2\times 3}a^{n-2}b^3 & +\text{ etc.}
\end{array}
$$

Hence,

$$(a-b)^{n+1}=a^{n+1}-\frac{(n+1)}{1}a^n b+\frac{(n+1)n}{1\times 2}a^{n-1}b^2-\frac{(n+1)(n)(n-1)}{1\times 2\times 3}a^{n-2}b^3+\text{etc.}$$

Examining this result, we see that the rules of § 166 give exactly the same terms as we have here.

From this we learn that *If the binominal theorem holds for the nth power, it holds also for the terms of the* $(n+1)$*th power ; that is, for the next higher power.*

Notice that we *assumed* the theorem held for the nth power, and *proved* that *if* it held for the nth power, it would hold also for the terms of the $(n+1)$th power.

To finish the demonstration we use mathematical induction. We say we know the theorem holds for the sixth power, for we got that result by actual multiplication. Therefore, by the demonstration just concluded, the theorem holds for the next higher, or seventh, power. Again, since it holds for the seventh power, by the theorem it holds for the eighth power; and so on indefinitely. Therefore, it holds for all powers.

287. To find the *r*th Term in the Binomial Development.

$$(a \pm b)^n = \overset{1}{a^n} \pm \overset{2}{\frac{n}{1}a^{n-1}b} + \overset{3}{\frac{n(n-1)}{1 \times 2}a^{n-2}b^2} \pm \overset{4}{\frac{n(n-1)(n-2)}{1 \times 2 \times 3}a^{n-3}b^3} + \text{etc.}$$

Here we have written the number of each term over it.

Suppose now we attempt to write down the *r*th term. To do this we notice that the last number subtracted from n in the numerator of any coefficient is always 2 less than the number of the term; that the last figure in any denominator is always 1 less than the number of the term; that the number subtracted from n in the exponent of a is always 1 less than the number of the term. Hence, the *r*th term in the value of $(a+b)^n$ is

$$\pm \frac{n(n-1)(n-2)\cdots(n-[r-2])}{1 \times 2 \times 3 \times \cdots r-1} a^{n-(r-1)}b^{r-1}.$$

1. Find the 6th term of $(a+b)^{13}$, by substituting in the formula for the *r*th term $n = 13$, $r = 6$.

SOLUTION. $\frac{13 \times 12 \times 11 \times 10 \times 9}{1 \times 2 \times 3 \times 4 \times 5}a^8b^5 = 1287\,a^8b^5$. *Ans.*

2. Find the 5th term of $(x+y)^{11}$.

3. Find the 4th term of $(x-y)^{17}$.

4. Find the 6th term of $(3x-2y)^{10}$. (See § 166, 5.)

5. Find the 12th term of $(2a-b)^{15}$.

288. Exercise in Raising Binomials to Powers by the Theorem.

1. Expand $(a+b)^{21}$, getting six terms of the answer.

2. Expand $(x^{\frac{2}{3}}+3)^6$. 3. Expand $(3x^{\frac{2}{3}}+a^{\frac{3}{2}})^7$.

4. Find the 7th term of $(2x-y)^{15}$.

5. Find $(2x-y)^{18}$ to seven terms in answer.

6. Expand $1-\frac{x}{3}$ to the 9th power.

7. Find the *r*th term of $(3x-2y)^{25}$.

8. Find the middle or 6th term of $\left(x - \frac{1}{x}\right)^{10}$.

9. Find the middle term of $(m^{\frac{1}{3}} - m^{-2})^8$.

10. Find the sum of the binomial coefficients for the 15th power.

Suggestion. If $(a + b)^{15}$ were expanded and then a and b each set equal to 1, we should get the sum of the coefficients, which sum evidently would equal left member, or 2^{15}.

11. Expand $\left(\sqrt{\frac{x}{2}} + \sqrt[3]{\frac{y}{3}}\right)^4$.

12. Expand $\left(\frac{\sqrt{-a}}{2} - \frac{\sqrt{-b}}{3}\right)^5$.

While the proof given in § 286 does not cover the case of fractional exponents, it is true that the binomial theorem holds good also for fractional and negative exponents.

13. Show that one gets the same result by applying the binomial theorem to $(a + b)^{\frac{1}{2}}$, as one does by extracting its root by the method of § 171.

14. Apply the binomial theorem to expand $(x + y)^{\frac{2}{3}}$ and $(x + y)^{\frac{7}{3}}$, and then form the product of these results, getting the first terms of $(x + y)^3$, or $x^3 + 3x^2y + 3xy^2 + y^3$.

15. Expand $(a + b)^{\frac{4}{3}}$ to four terms.

16. Expand $(x - 2)^{-2}$ to five terms.

17. Expand $(a - b)^{-\frac{5}{6}}$ to four terms.

CHAPTER XIII

INEQUALITIES

289. Inequalities. An **inequality,** or an **inequation,** is a statement that one quantity is greater or less than another. The sign for "greater than" is $>$, and that for "less than" is $<$. The opening of the angle is towards the greater quantity. These symbols, with $=$, are called symbols of relation. By "greater than" is meant higher up in the algebraic scale.

290. Principles governing the Solution of Inequations.

1. *It is allowable to* **transpose** *in an inequality just as in an equation.*

PROOF. Let $a \pm c > b$.

Then, $a > b \mp c$. (Equals added to or subtracted from unequals give unequals, since to do this merely slides both members of the inequation the same distance along the algebraic scale.)

2. *It is allowable to multiply or divide an inequation by the same positive number; but multiplying or dividing by a negative number reverses the sign.*

PROOF. (1) Let $a > b$, and m be a positive number.

Then, $ma > mb$. (Unequals multiplied by the same positive number give unequals, and that product is the greater which is obtained from the greater of the two unequals. This is true whatever signs a and b have.)

(2) Let m be negative.

Then, $ma < mb$. (By the preceding case and because the quantity which was greater than the other *at first* becomes less than the other *afterwards*, when the signs of both are changed.)

The pupil should test these statements with numbers expressed in the Arabic notation.

3. *Inequations can be added, greater to greater and less to less, the former sum being greater than the latter, but it is not allowable to subtract one inequation from another.*

The first part of the theorem is evidently true, but the student may not see why he cannot subtract one inequation from another. An example or two is sufficient here, since if the rule fails in *one* instance, it does not hold.

Thus,
$$\begin{array}{r} 12 > 10 \\ \underline{\ \ 5 > \ \ 4} \\ 7 > \ \ 6 \end{array} \qquad\qquad \begin{array}{r} 12 > 11 \\ \underline{\ \ 5 > \ \ 2} \\ 7 \ngtr \ \ 9 \end{array}$$

4. *The members of an inequation both of whose terms are positive can be raised to the same power.*

291. Special Theorems in Inequations, and Exercises.

1. To prove that the sum of the squares of any two real and unequal numbers is greater than twice their product.

Proof. Let a and b be any two real and unequal numbers.

Then, $(a-b)^2 > 0$. (Since a square number is always positive, that is, greater than 0.)

Or $a^2 - 2ab + b^2 > 0$. (Squaring $a - b$.)

$\therefore a^2 + b^2 > 2ab$. Q.E.D. (§ 290, 1.)

2. Prove that any positive fraction whose terms are unequal plus its reciprocal is greater than 2.

Thus, to prove $\frac{x}{y} + \frac{y}{x} > 2.$

Suggestion. Make this Ex. depend on Ex. 1, using § 290, 2.

3. Prove that the arithmetic mean of two different positive numbers is greater than the geometric mean.

Thus, to prove $\frac{a+b}{2} > \sqrt{ab}.$

Suggestion. Square both members, § 290, 4.

4. Show that $x^3 + y^3 > x^2y + xy^2$ when $x + y > 0$. (See § 290, 2).

5. Show that $x^3 + 1 > x^2 + x$ when $x + 1 > 0$ and $x \neq 1$.

6. Show that $x^2 + y^2 + z^2 > xy + xz + yz$ if x, y, z are unequal.

SUGGESTION. Use § 291, 1, and § 290, 3, and then divide through by 2.

7. If $2x + \frac{1}{2}x - 4 > 6$, show that $x > 4$.

8. Show that the difference between two sides of a triangle is less than the third side. If a, b, c, are the three sides, show that $c - b < a$, using § 290, 1.

9. Two of the sides, a and b, of a triangle are respectively 9 and 15 inches. Between what limits must the third side c lie?

10. If $3x - 7 - 5x > \frac{3}{2}x - 9$, what limiting value does x have?

11. If a, b, and c are positive numbers, and if $a > b$, prove that $\frac{a+c}{b+c} < \frac{a}{b}$.

12. For what values of x is $x^2 - 6x - 7 > 0$? (See § 210.)

13. A's record time for running a mile is 4 minutes and 30 seconds, and B's is 4 minutes and 40 seconds. Less than how many feet handicap shall A give B so as still to beat him?

CHAPTER XIV

EXERCISES IN PHYSICS*

292. The Value of Algebraic Notation in Physics. To see the value of the algebraic notation take, for example, the familiar formula $s = \frac{1}{2} gt^2$. If we translate this formula into words, we must say, "the distance through which a body will fall from rest under the influence of gravity equals the product of one half of the acceleration of gravity (or increase of velocity per second) and the square of the number of seconds."

Now when one becomes accustomed to it, he can think distance easier and more quickly from seeing s than from seeing the words "distance a body falls through in t seconds," and he can think number of seconds elapsed quicker from seeing t than from seeing the corresponding words. This explains in part why the algebraic notation is so valuable in all kinds of scientific work.

293. Exercise in solving Formulas from Mechanics. In the following formulas $s =$ distance, $v =$ velocity, $t =$ time, $g =$ acceleration of gravity or increase of velocity per second.

1. The formula for uniform motion is $s = vt$. Solve this equation first for v and then for t. *Ans.* $v = \frac{s}{t}$. (Division Axiom.)

2. The formulas for uniformly accelerated motion are

$$(1)\ v = gt. \qquad (2)\ s = \tfrac{1}{2}gt^2.$$

Solve (1) for t and then for g. Next solve (2) first for t and then for g. Next eliminate t between (1) and (2) and solve the resulting equation for each letter in it.

3. Solve $fs = \frac{1}{2} mv^2$, where f is a force and m is the mass (weight) of a body, for each letter in it.

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* If the class has not studied physics, this chapter should be omitted.

4. *Projectiles.* Formulas for projectiles fired vertically are

(1) $v = u - gt.$ (2) $s = ut - \frac{1}{2} gt^2.$

In these formulas u denotes the constant velocity caused by the initial impulse.

Solve (1) for u, g, and t in turn. Similarly solve (2) for u, g, and t in turn. Notice that solving for t calls for the solution of a complete quadratic. *Ans. to last,* $t = \frac{u \pm \sqrt{u^2 - 2gs}}{g}$.

5. Find s, or the distance a body falls in 5 seconds, by substituting in $s = \frac{1}{2} gt^2$. Take $g = 32.15$.

6. A stone fell from a balloon a mile high. How many seconds elapsed before it reached the earth?

7. A bullet is shot upward with a velocity of 2000 ft. a second. How high will it rise?

SUGGESTION. Substitute in the formula obtained from $v = u - gt$ and $s = ut - \frac{1}{2} gt^2$ by eliminating t, setting $v = 0$ for the velocity at the highest point.

8. A stone is dropped into a shaft 1029.12 ft. deep and the sound of its striking the bottom is heard after 8.9 sec. Find the velocity of sound.

SUGGESTION. Let x = number of seconds spent in falling 1029.12 ft. Solve for x. How many, then, were spent in the sound returning? What, then, is the velocity of sound? Use $g = 32.16$.

9. A stone is *thrown* down a shaft 457.2 ft. deep with an initial velocity of 50 ft. per second. How many seconds will it take to reach the bottom? Use $s = ut + \frac{1}{2} gt^2$, with $-$ changed to $+$.

10. From the equations $E = FS$, $F = ma$, $S = \frac{1}{2} at^2$, and $v = at$, find E in terms of m and v.

11. Solve $F = \frac{4\pi^2 mr}{t^2}$ for m, r, and t. (Centrifugal force.)

12. Solve $FS = \frac{1}{2} MC(V_1^2 - V^2)$ for F, M, and V.

13. A person in an aëroplane dropped a stone to the ground and a person on the earth counted with a stop watch the number of seconds it took to fall, and found the time 12 seconds. How high was the aëroplane as given by the formula $s = \frac{1}{2}gt^2$ (which does not allow for the resistance of the atmosphere), if $g = 32.16$?

294. General Exercise in the Solution of Physics Problems.

1. If air is contained in a vessel of volume V under pressure of p pounds per square inch, Boyle's law states that the product of volume V and pressure p equals a constant number c. Express this truth algebraically.

2. Make a formula for the "gear" of a bicycle, if r is number of inches in radius of rear wheel, l = number of cogs in large sprocket wheel, and s = number of cogs in small sprocket.

Suggestion. — The gear of a bicycle is the diameter in inches of a high wheel one revolution of which would cover as much ground as a safety bicycle does with one revolution of the pedals. If there were 21 cogs in large sprocket and 7 in small sprocket, then rear wheel will turn three times round while pedals turn once.

3. Find the gear of a bicycle from the formula made if $r = 14$, $l = 24$, $s = 9$.

4. Two electric lamps of 8 and 16 candle power, one meter apart, illumine equally two sides of a photometer screen in line with them. If the illumination is inversely as the square of the distance of the light, how many meters is the 8 candle-power lamp from the photometer screen?

Suggestion. If n = number of meters required, then (§ 310)

$\frac{8}{n^2} = \frac{16}{(n+1)^2}$. (Since the quantity of light is proportional to candle power.)

5. Suppose in Ex. 24 that the lamps were of 8 and 32 candle-power, and one lamp was 2 meters farther from photometer screen than the other. How far was the 8 candle-power lamp from the photometer screen?

6. The number of vibrations n of a stretched string per second is given by the formula

$$n = \frac{1}{2l}\sqrt{\frac{Mg}{m}}$$

where $l =$ length of string in centimeters, $M =$ pull on string in grams, $m =$ weight of 1 cm. of string in grams, and $g = 980$. Find n when $M = 5000$, $l = 68.2$, and $m = .005$.

7. Number of horse power (H.P.) is given by the formula

$$\text{H.P.} = \frac{\text{Number of foot pounds}}{33000 \times \text{number of minutes}},$$

in which one foot pound means the equivalent of 1 lb. lifted vertically 1 ft. Find operating H.P. when elevator weighing 550 lb. is raised 75.5 ft. in 2 min.

8. Kinetic energy (K.E.) is given in foot pounds by the formula

$$\text{K.E.} = \frac{Wv^2}{2g},$$

in which $W =$ weight in pounds, $v =$ velocity in feet per second, and $g = 32.16$. What is the kinetic energy of a pile driver falling with a velocity of 46.7 ft. per second if it weighs 112.5 lb.?

9. If $F = \frac{mm'}{(r-l)^2} - \frac{mm'}{(r+l)^2}$, and l^2 is so small that it can be set equal to zero, show that $F = \frac{4\,mm'l}{r^3}$.

10. If d, v, w, $(w = dv)$ represent the density, volume, and weight of one kind of metal, and d', v', and w the density, volume, and weight of an equal weight of another, find v' in terms of d, v, and d'.

Two pieces of metal, one of iron and the other of copper, weigh the same number of grams. The iron has a volume of 250 cu. cm. What is the volume of the copper if the density of iron is 7.6 and that of copper 8.8? Solve by substituting in the formula just found.

11. The volumes of two bodies are v and v' and their specific gravities are s and s' respectively. Show that the specific gravity

of the mixture equals $\frac{vs + v's'}{v + v'}$ provided that no condensation exists, *i.e.* provided that the volume of the mixture equals the sum of the volumes of the ingredients.

SUGGESTION. Specific gravity × volume = c × weight, where c is a constant. Let x = specific gravity of mixture. What is weight of one ingredient? of the other? of the mixture? What then is the specific gravity of the mixture?

12. The air is a mixture of 21 parts of oxygen and 79 of nitrogen. If the specific gravity of oxygen is 1.1026, what is the specific gravity of nitrogen?

SUGGESTION. Here $v = 21$, $v' = 79$, and $s = 1$, air being the standard substance.

13. When two bodies whose specific gravities are s and s' are combined, what is the specific gravity of the mixture or combination if their absolute weights are w and w'?

SUGGESTION. Write $sv = cw$, $sv' = cw'$, and substitute in formula, Ex. 11.

Ans. $\frac{(w + w')ss'}{(ws' + w's)}$.

14. What is the specific gravity of a coin composed of 9 parts by weight of silver and 1 of copper, if the specific gravity of silver is 10.5 and that of copper 8.8?

15. A gas expands $\frac{1}{273}$ of its zero Centigrade volume for each degree of rise of temperature. To find the volume of a gas, being given its volume at freezing and temperature.

Let V_0 = volume of a gas at zero degrees Centigrade, *i.e.* at freezing,

V_t = volume of the same gas at $t°$.

Then,
$$V_t = \left(1 + \frac{t}{273}\right)V_0.$$

To what volume will 1 cu. ft. of air at 0° Centigrade increase when the temperature rises 48° if the pressure remains constant?

16. To what volume will 20 cu. ft. of air contract when the temperature falls from 98° to 12° Centigrade?

SUGGESTION. Substitute 98 and 12 in the formula and eliminate V_0.

17. The attraction of gravitation between two bodies varies *inversely* as the square of their distances apart. If F is the force when they are D miles apart, and f the force when they are d miles apart, then $F:f = \frac{1}{D^2} : \frac{1}{d^2}$, or $\frac{F}{f} = \frac{d^2}{D^2}$.

Forces can be measured by the distance they cause a body to move in 1 second. A body at the surface of the earth, 3963 miles from the earth's center, is made to move 16.08 ft. in one second. Knowing the distance of the moon from the center of the earth, we have three terms of the proportion $F:f = d^2 : D^2$. Thus, we have given $f = 16.08$ ft., $d = 3963$ mi., $D = 238{,}850$ mi., to find F or the distance the moon is pulled by gravity in one second. Solving the proportion for F, $F = \frac{1}{228}$ ft.

Knowing the mean distance of the moon from the earth, and the time of its revolution around the earth, it is a simple numerical calculation to find that the moon in one second is pulled $\frac{1}{228}$ ft. away from the straight line in which it would otherwise move if it were not attracted by the earth.

Note. Sir Isaac Newton found in the year 1684 that if he substituted in the formula $F:f = d^2 : D^2$ the accepted values of f, d, and D he got the distance the moon is pulled each second. It is said that he was kept from announcing his discovery of the law of gravitation eighteen years earlier, because the data used were incorrectly given at the earlier period.

18. A barometer tube l millimeters in length closed at lower end and open at upper end is filled with mercury to a height of h millimeters. Closing the upper end with the thumb, reverse the tube and put open end down into a dish of mercury. If the outside air pressure is b millimeters, to what height x will the mercury in the tube move to, counting from the level of the dish?

Suggestion. Evidently at first the pressure in the tube is b millimeters. If the column of mercury in the tube at first is under a pressure of b millimeters from the top of the tube, the mercury will drop down until the lessened pressure in the tube + weight of mercury still in tube = pressure of air outside. Use Boyle's law to get new pressure.

Ans. $x = \frac{b+l}{2} \pm \sqrt{\left(\frac{b+l}{2}\right)^2 - bh}.$

19. Given the length of a barometer tube as 840 mm., $h = 250$, and $b = 760$. Find the height the mercury takes.

CHAPTER XV

GENERALIZATION — INDETERMINATE EQUATIONS — NEGATIVE SOLUTIONS

I. GENERALIZATION

295. Generalization. The elementary algebra *generalization* of a problem is attained by replacing its known numbers by letters and then deriving its solution. Problems are studied using particular numbers; afterwards, on account of the frequency with which they occur, they are generalized. Then to work a problem one merely substitutes the particular numbers for the letters in the generalized solution. Generalization is one of the most important topics in algebra. Indeed, it is the very essence of the subject.

Historically the steps in the development of the idea of generalized number may have been somewhat as follows: First probably came the use of a letter to denote an unknown number. This was followed by the use of a letter to denote any positive number. Then came the notion of allowing several letters to enter into a problem, some known and some unknown. The use of a letter to denote a number which might be either positive or negative, came later than might be supposed from our familiarity with the conception. We have also the more general use of a letter which allows it in a formula to stand for any combination of other letters and signs. Last of all, perhaps, may be mentioned the use of a letter to stand for some complex concept no longer a mere quantity.

1. In § 146, we solved problems like this:

$$\begin{cases} (1)\ 3x + 4y = 25, \\ (2)\ 5x - 2y = 7. \end{cases}$$

They were all solved in very much the same way. The question arises, Can we not work *one* problem with letters for coefficients which, since a letter may bear any value, will include all the others? The answer is, yes, we can. This was exactly what was done in § 160, 1, when we solved $\begin{cases} (1) & ax + by = c, \\ (2) & a'x + b'y = c'. \end{cases}$

From that solution we got $x = \dfrac{b'c - bc'}{ab' - a'b}$, $y = \dfrac{ac' - a'c}{ab' - a'b}$.

$$\text{Comparing } \begin{cases} 3x + 4y = 25, \\ 5x - 2y = 7, \end{cases} \text{ with } \begin{cases} ax + by = c, \\ a'x + b'y = c', \end{cases}$$

we see that $a = 3$, $b = 4$, $c = 25$; $a' = 5$, $b' = -2$, $c' = 7$.

Substituting these values in the values of x and y, above, we get

$$x = \frac{(-2) \times 25 - 4 \times 7}{3 \times (-2) - 5 \times 4} = 3; \quad y = \frac{3 \times 7 - 5 \times 25}{3 \times (-2) - 5 \times 4} = 4.$$

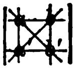
VERIFICATION. $3 \times 3 + 4 \times 4 \equiv 25$; $5 \times 3 - 2 \times 4 \equiv 7$.

2. Solve the following problems by substituting values in the generalized solution of § 160, 1, as was done in the preceding exercise, and verify the answers found:

$$\begin{cases} 3x + 2y = 23, \\ x + y = 8. \end{cases} \qquad \begin{cases} 5x + y = 12, \\ x + 5y = 36. \end{cases} \qquad \begin{cases} 5x + 7y = 32, \\ 7x - 3y = 32. \end{cases}$$

Any number of such exercises can be had from §§ 146, 147.

3. *"Determinant" notation for the values of x and y in* **1** *above.* In what is called the determinant notation the rule for writing down these values is simple. In determinants the quantities involved are written in a *square* between two vertical lines.

Thus, $ab' - a'b$ is written $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$. In , we have two horizontal *rows*, two vertical *columns*, and two *diagonals*, namely, the *principal* diagonal ab', and the *secondary* diagonal $a'b$.

4. *The notation,* $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$, *signifies the product of the two quantities in the principal diagonal less the product of the two quantities in the secondary diagonal.*

Thus, in 1, of this article,

$$x=\frac{\begin{vmatrix} c & b \\ c' & b' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}}=\frac{b'c-bc'}{ab'-a'b};\quad y=\frac{\begin{vmatrix} a & c \\ a' & c' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}}=\frac{ac'-a'c}{ab'-a'b}.$$

Comparing the determinant values of x and y with the equations $\begin{cases} ax+by=c, \\ a'x+b'y=c', \end{cases}$ we see *that for the denominator of each we merely take the four coefficients of x and y and write them, as they stand, in the form of a determinant; and for the numerator of x, we take its denominator and substitute c, c' for the coefficients of x; and for the numerator of y, we take its denominator and substitute c, c' for the coefficients of y.*

5. Solve $\begin{cases} 4x-7y=-2, \\ 5x+3y=21, \end{cases}$ by determinants.

Solution. $$x=\frac{\begin{vmatrix} -2 & -7 \\ 21 & 3 \end{vmatrix}}{\begin{vmatrix} 4 & -7 \\ 5 & 3 \end{vmatrix}}=\frac{-6-(-147)}{12-(-35)}=\frac{141}{47}=3. \quad \textit{Ans.}$$

$$y=\frac{\begin{vmatrix} 4 & -2 \\ 5 & 21 \end{vmatrix}}{\begin{vmatrix} 4 & -7 \\ 5 & 3 \end{vmatrix}}=\frac{84-(-10)}{12-(-35)}=\frac{94}{47}=2.$$

Verification. $4\times 3-7\times 2\equiv -2$; $5\times 3+3\times 2\equiv 21$.

Solve the following by the determinant formulas, verifying:

6. $\begin{cases} 4x+9y=3, \\ 3x+7y=2. \end{cases}$ 7. $\begin{cases} 2x+7y=11, \\ 5x-9y=1. \end{cases}$ 8. $\begin{cases} 5x-3y=20, \\ 3x-4y=1. \end{cases}$

9. Other problems can be had by using those in §§ 146, 147. A little practice will enable the student to get the answers very quickly. If the coefficients are small, the problems can be solved mentally.

Remark. The determinants we have been using are said to be of the second order. One of the third order consists of three rows and three columns of numbers, as,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Such determinants can be used to solve simultaneous equations like those in § 149, but the solutions are too complicated for this course.

10. Generalization of the problem in which the *sum* and *difference* of two numbers are given to find the numbers.

We will first solve a number of special cases and will then generalize the solution.

Find two numbers whose sum is 168 and difference 34.

Solution. Let $x =$ the first number, and $y =$ the second number.

Then,

$$\begin{array}{rl} x + y &= 168 \\ x - y &= 34 \\ \hline 2x &= 202 \text{ (Add. Ax.)} \\ x &= 101. \end{array} \qquad \begin{array}{rl} x + y &= 168 \\ x - y &= 34 \\ \hline 2y &= 134 \text{ (Sub. Ax.)} \\ y &= 67. \end{array}$$

Verification. $101 + 67 = 168$; $101 - 67 = 34$.

Solve in the same way the following: Find two numbers when their sum is 538 and difference is 162; also when their sum is 793 and their difference 259; also when their sum is 961 and their difference 148; also when their sum is 9.643281 and their difference 9.043216.

To generalize this problem denote sum by s and difference by d. Then

$$\begin{array}{rl} x + y &= s \\ x - y &= d \\ \hline 2x &= s + d \\ x &= \dfrac{s + d}{2}. \end{array} \qquad \begin{array}{rl} x + y &= s \\ x - y &= d \\ \hline 2y &= s - d \\ y &= \dfrac{s - d}{2}. \end{array}$$

From these formulas we have the *theorem: The greater of two numbers equals half the sum of their sum and difference; and the less equals half the difference between their sum and difference.*

11. Given the sum of two numbers as 324 and their difference as 62, to find the numbers *by substituting the values in the formula.*

12. Generalize: Divide \$183 between two men so that $\frac{4}{7}$ of what the first receives shall equal $\frac{3}{10}$ of what the second receives; then solve it *by substitution in the formula found.*

SOLUTION. By the definition of generalization at the head of this article, each particular number of the problem is replaced by a letter. Making this change (choosing any convenient letters), the problem comes to read: Divide \$$a$ between two men so that $\frac{m}{n}$ of what the first receives shall equal $\frac{p}{q}$ of what the second receives.

Let $x =$ no. of dollars the first man got,

and $a - x =$ no. of dollars the second man got.

Then by the conditions of the problem,

$$\frac{m}{n}x = \frac{p}{q}(a - x).$$

$$\frac{mx}{n} = \frac{ap - px}{q}. \qquad \text{(Multiplying.)}$$

$$mqx = anp - npx. \qquad \text{(Mult. Ax.)}$$

$$mqx + npx = anp. \qquad \text{(Unknowns to left member, Ax. ?)}$$

$$x = \frac{anp}{mq + np}. \qquad \text{(Factoring, and Ax. ?)}$$

Now, comparing the general problem with the given one, we see that $a = 183$, $m = 4$, $n = 7$, $p = 3$, $q = 10$. Substituting these values in

$$x = \frac{anp}{mq + np},$$

we get, $x = \frac{183 \times 7 \times 3}{4 \times 10 + 7 \times 3} = \frac{183 \times 7 \times 3}{61} = 63$, the 1st no.

Then, $183 - 63 = 120$, the 2d no.

VERIFICATION. $\frac{4}{7}$ of $63 \equiv \frac{3}{10}$ of 120.

296. Exercise in the Generalization of Problems. Let a take the place of the first known number, b take the place of the second, and so on.

For problems to generalize we use those of § 140. After solving, substitute the particular values of the problem generalized.

1. What number is that whose ath part and bth part and c more equals d times the number and e more? See § 140, 1.

$$\frac{x}{a} + \frac{x}{b} + c = dx + e.$$

$$bx + ax + abc = abdx + abe. \quad \text{(Ax. ?)}$$

$$bx + ax - abdx = abe - abc. \quad \text{(Ax. ?)}$$

$$(a + b - abd)x = ab(e - c). \quad \text{(Factoring.)}$$

$$x = \frac{ab(e-c)}{a+b-abd}. \quad \text{(Ax. ?)}$$

Check. When we generalized we put a for 2, b for 3, c for 100, d for 2, and e for 2. We will now substitute these values in the answer just found.

$$x = \frac{2 \times 3\,(2 - 100)}{2 + 3 - 2 \times 3 \times 2} = 84. \quad \text{(See § 21.)}$$

Verification of 84 as answer. $\frac{1}{2}$ of $84 + \frac{1}{3}$ of $84 + 100 = 2 \times 84 + 2$.

2. What number is that whose ath part, bth part, and cth part together are equal to d? See § 140, 2. Check and verify as in the preceding solution.

3. Generalize now in the same way some of the problems of § 140, § 150.

Check and then verify as in the solution above.

Remark. If the student finds himself unable to proceed with the making of the statement of a generalized problem, he will probably be much aided if he will first solve the problem as it stood before it was generalized.

II. INDETERMINATE EQUATIONS

297. Indeterminate Equations. The student has probably noticed that so far, except in graphs, we have always had as many equations given as there were unknowns in the equations. Thus, when there were two unknowns, two equations were given; when there were three unknowns, three equations were given; and so on.

When one equation is given containing two unknowns, a *graph* can be constructed from it.

If the number of given equations is less than the number of unknowns, we have what are called **indeterminate equations.** These equations can have an indefinite number of values of x and y. Thus, we can assign x *any* value and get a corresponding value of y. They are solved sometimes for *integral* values of the unknowns, *but the method of solution is altogether different* from the solution of determinate equations. We will solve a few of these.

1. Separate 71 into two parts, one of which is divisible by 5 and the other by 8 without remainders.

Solution. Let $5x$ = one part, and $8y$ = other part.
Then $5x + 8y = 71$ is the only equation obtainable.
Solving, we get $x = \frac{71 - 8y}{5}$.

Now, by the conditions of the problem, x is a whole number. Then

$$\frac{71 - 8y}{5}, \text{ or } 14 - y + \frac{1 - 3y}{5} \text{ (§ 104) is a whole number.}$$

But as y must be a whole number, then $14 - y$ is a whole number, and then also $\frac{1 - 3y}{5}$ must be a whole number. Then $2 \times \frac{1 - 3y}{5} = \frac{2 - 6y}{5}$ is a whole number. (We multiply by such a number that when we divide by the denominator, the coefficient of y in the remainder is 1). Then $-y + \frac{2 - y}{5}$ (§ 104) is a whole number. In order that $\frac{2 - y}{5}$ shall be zero or a whole number, $y = 2$, or 7, or 12, or 17, etc. Substituting these values of y in the original equation, we find for $y = 2$, $x = 11$; for $y = 7$, $x = 3$; for $y = 12$, $x = -5$. The last value of x is regarded as inadmissible here, as would be other values of x obtained from subsequent values of y. Hence, $5x = 55$, $8y = 16$, and $5x = 15$, $8y = 56$ are the values sought.

Check. Draw graph of equation; find pairs of integral coördinates on it.

2. Separate 97 into two parts, one of which is divisible by 7 and the other by 9.

3. A woman paid $106 for silk at $2.50 a yard, and velvet at $3.50 a yard. How many yards of each did she buy?

4. A person bought 40 animals, consisting of pigs, geese, and chickens, for $40. The pigs cost $5 apiece, the geese $1, and the chickens 25¢ each. Find the number of each bought.

Suggestion. In this problem we get two equations and have three unknowns. After eliminating one of the unknowns between the two given equations, proceed with the resulting equation as in (1) above.

5. A farmer buys oxen, sheep, and hens. The number bought is 100, and the total cost $100. If the oxen cost $35, the sheep $3, and the hens 25¢ each, how many of each did he buy?

III. NEGATIVE SOLUTIONS

298. Negative and Inconsistent Solutions.

1. If from $\frac{5}{8}$ of a certain number 1 is subtracted, the remainder equals $\frac{3}{4}$ of the number. *Ans.* $x = -8$.

The negative answer shows that there is some arithmetical inconsistency in the problem. Evidently if we take something from $\frac{5}{8}$ of a positive number, we cannot have $\frac{3}{4}$ of the number (which is greater than $\frac{5}{8}$ of it) left.

2. A father's age is 40 years, and his son's age is 13; in how many years will the father's age be 4 times that of the son? *Ans.* -4 yr.

This negative answer shows that the problem should have read "how many years *ago*" instead of "in how many years."

3. A grocer had two casks of oil, one of which contained twice as much as the other. From the larger cask he drew 16 gal., and from the smaller 10 gal. Then after a fourth of what remained in the larger cask had been withdrawn, the two casks were found to contain an equal number of gallons. How much did each cask hold? *Ans.* 8 gal. and 4 gal.

That these answers are absurd, though obtained without error in the solution, is seen the moment one tries to verify. We learn from the preceding solutions the following:

4. *That a negative result may indicate either arithmetical incongruity in the statement of the problem, or at least that the number obtained as the answer should be taken in a sense contrary to that implied in the statement of the problem.*

5. *That positive results do not necessarily prove that problems can be solved.* The negatives may have entered in the process of solution, and then dropped out again.

6. *If the given concrete quantities spoken of in the statement of the problem are such that they can have either positive or negative values, then the result can be interpreted; otherwise it may not be capable of interpretation.*

CHAPTER XVI

REASONING IN EQUATIONS — DISCUSSION OF THE QUADRATIC — DISCUSSION OF PROBLEMS — DEFINITIONS

I. REASONING IN EQUATIONS

299. Validity of Processes in the Solution of Equations. It might be supposed that any root of any equation obtained by axiomatic processes would verify when substituted in the original equation, but such is not always the case. This does not mean that the axioms are not true, but that new equations obtained by means of them are not **equivalent** to the given equation, that is, do not have identically the same roots as the given equation.

300. Equivalent Equations. *Reversibility of the steps of a solution.* Whether a root obtained will satisfy the given form of the equation depends on whether the steps of the solution are *reversible.* Two equations such that either can be derived from the other, and which have exactly the same roots are said to be **equivalent** (see p. 110). The processes used in the solution of *simple* equations rarely give anything except equivalent equations, but in the solution of radical, quadratic, and higher equations, new roots are frequently introduced, and sometimes roots of the given equations are lost in the process of solution.

1. *Two equations are equivalent if one can be obtained from the other by adding the same quantity to both members.* (By Addition and Subtraction Axioms.)

PROOF. Let S and S' denote the two members of an equation and a any third quantity, each of these being capable of containing both known and unknown quantities.

Then, any value of the unknown which satisfies $S = S'$ evidently satisfies $\boldsymbol{S}+\boldsymbol{a}=\boldsymbol{S}'+\boldsymbol{a}$. Also any value which satisfies $\boldsymbol{S}+\boldsymbol{a}=\boldsymbol{S}'+\boldsymbol{a}$, also satisfies $\boldsymbol{S}=\boldsymbol{S}'$.

Since transposition depends on the addition and subtraction axioms, equations obtained by it are equivalent.

2. *Equations obtained by multiplying or dividing both members of an equation by the same known quantity are equivalent.*

If a is a known number, and if $\boldsymbol{S}=\boldsymbol{S}'$, then $a\boldsymbol{S}=a\boldsymbol{S}'$; also if $a\boldsymbol{S}=a\boldsymbol{S}'$, then $\boldsymbol{S}=\boldsymbol{S}'$.

3. *An equation obtained from another by squaring its members or raising them to a higher power, or extracting the same root of each member is not equivalent to the given equation.*

As before, let $\boldsymbol{S}$ and $\boldsymbol{S}'$ denote the two members.

We have $\boldsymbol{S}=\boldsymbol{S}'$,

whence $\boldsymbol{S}^2=\boldsymbol{S}'^2$; (Sq. Ax.)

then $\boldsymbol{S}^2-\boldsymbol{S}'^2=0$. (Sub. Ax.)

Factoring, $(\boldsymbol{S}+\boldsymbol{S}')(\boldsymbol{S}-\boldsymbol{S}')=0$.

We have then $\boldsymbol{S}+\boldsymbol{S}'=0$, whence $\boldsymbol{S}=-\boldsymbol{S}'$; (Solution by factoring.)

$\boldsymbol{S}-\boldsymbol{S}'=0$, whence $\boldsymbol{S}=\boldsymbol{S}'$.

The last equation is identical with the given equation, but we have also the equation $\boldsymbol{S}+\boldsymbol{S}'=0$ *in addition*, which is a new equation.

EXAMPLE. Given $\sqrt{x+5}+\sqrt{x}=1$.

$\sqrt{x+5}=1-\sqrt{x}$. (Sub. Ax.)

$x+5=1-2\sqrt{x}+x$. (Sq. Ax.)

$2\sqrt{x}=-4$. (Sub. Ax.)

$4x=16,\ x=4$. (Sq. Ax.)

VERIFICATION. $\sqrt{4+5}+\sqrt{4}\neq 1$, or, $3+2\neq 1$.

Thus 4 is not a root of the given equation. Notice that if we had started with the equation $\sqrt{x+5}-\sqrt{x}=1$, the solution would have differed from that above only in a few signs, and the answer would have been 4, which answer would have verified. The three equations $\sqrt{x+5}+\sqrt{x}=\pm 1$, and $-\sqrt{x+5}+\sqrt{x}=1$ have been called impossible equations, since if the signs are taken just as they stand there is no root that will satisfy them.

4. ***An equation may not be multiplied or divided through by a function of x which equals 0, or some number divided by 0, when the value of x is substituted in it.***

Example.

$$\frac{y+\frac{1}{y}}{y-\frac{1}{y}}+\frac{1+\frac{1}{y}}{1-\frac{1}{y}}=2.$$

$$y+\frac{1}{y}-1-\frac{1}{y^2}+y+1-\frac{1}{y}-\frac{1}{y^2}=2\left(y-1-\frac{1}{y}+\frac{1}{y^2}\right). \quad \text{(Mult. Ax.)}$$

Clearing this equation of fractions and collecting, we get

$$4y^2+4y=8, \text{ whence } y=1 \text{ or } -2.$$

But while $y=-2$ satisfies the original equation, $y=1$ *does not*, since $\frac{2}{0}+\frac{2}{0}$ on the left side *has no meaning*, if 0 is understood to be what is called absolute 0, and not the limit of some quantity, while 2 on the right side is 2. The reason why the root $y=1$ appeared is because when we first cleared of fractions, we multiplied the equation by $y-1$, and because $y=1$ we were really multiplying by 0, which is not allowable by the principle just stated.

This principle explains the fallacy in the so-called "proof" that $1=2$.

Thus, let $a=x$; then $ax-x^2\equiv a^2-x^2$;

whence, on factoring, $x(a-x)=(a+x)(a-x)$.

Then $x=a+x$, (Dividing by $a-x$.)

or, $x=x+x=2x$; (Since $a=x$.)

whence $1=2$. (Dividing by x.)

The first and second equations are true equations; the third and all the rest are false, since to get the third we divided by $a-x$, and as $a=x$ making $a-x=0$, we were dividing by zero.

We may set forth the truth, illustrated under 4, more briefly as follows:

Let R be any expression containing the unknown, and, as before, let S and S' be the two members of an equation.

Then, $S=S'$.

Whence $RS=RS'$. (Mult. Ax.)

Then, $RS-RS'=0$, or $R(S-S')=0$. (Transposing and factoring.)

Setting each factor equal to 0, we have $R=0$, and $S-S'=0$.

Thus, the new factor with its root has been introduced into the equation.

This case includes 3, above, since we can write $\boldsymbol{R} = \boldsymbol{S} + \boldsymbol{S}'$.

In conclusion, then, if at any time we cannot tell whether a root found belongs to the given equation, we always have this recourse that we can substitute it in that equation and find out. Roots should always be verified in the original equations.

301. The Analytic Method. Algebra uses the *analytic* method; arithmetic, the *synthetic*. When a problem is solved by the synthetic method, one starts with the numbers that are given, and, by combining them in a certain order, obtains the answer. In algebra, on the other hand, one treats the answer as if known, calling it x; then, forming the equation, the course is *backward* to the answer.

To illustrate, take the problem, What number is that whose square increased by 11 times the number equals 60? Putting this in the form of an equation, we have

$x^2 + 11x = 60.$ (If there is such a number as described in the problem, then this equation is true.)

Then, $4x^2 + 44x = 240.$ (If the preceding equation holds, this one holds.)

$4x^2 + 44x + 121 = 361.$ (If the preceding equation holds, this one holds.)

$2x + 11 = \pm 19.$ (If the preceding equation holds, this one holds.)

$x = 4, \text{ or } -15.$ (If the preceding equation holds, this one holds.)

Now, we assume we can reverse the reasoning. We know there is such an equation as $x = 4$, or $x = -15$; then (multiplying each member by 2, and adding 11 to each result), the fourth equation holds, then the third, then the second, then the first. In this way we see that the steps of our solutions should be reversible if we desire to draw safe conclusions.

The problem 'to find a number such that the sum of its square root and the square root of 5 more than the number is unity' gives the equation of the preceding article, $\sqrt{x+5}+\sqrt{x}=1$. By solving this equation, 4 is found for the value of x. But we are prepared to find that 4 does not satisfy the given condition, because one of the steps of the solution (squaring both members) was not reversible.

302. Exercise in Reasoning in Equations.

1. Prove that cubing the members of an equation introduces a new factor with new roots.

2. Solve $2x\sqrt{x^2+3}-2x\sqrt{x^2+2}=1$, and verify.

3. Solve $\frac{x-3}{5}-\frac{8}{x^2}=\frac{2}{x}-\frac{8-6x+x^2}{x^2}$, and verify all the roots.

4. Solve $\frac{5x-a}{x+a}+\frac{x}{x+b}=\frac{4x-2a}{x+a}+\frac{4x-a}{x+b}+1$. Verify.

5. Solve $2\sqrt[3]{x^3+5x+19}=(x+3)(\sqrt{-3}-1)$. Verify.

SUGGESTION. Solve also same equation with $x+3$ as right member.

303. Systems of Equations. We saw in §§ 147, 148, that a set of two given equations might be replaced by one new equation and one of the given equations, or by two entirely new equations.

An equation obtained by adding the members of two equations, or by adding after each equation has been multiplied through by a known quantity, can take the place of either of the given equations.

Thus, if $S=S'$ and $T=T'$, then $S+T=S'+T'$;
also if $S+T=S'+T'$, and $S=S'$, then $T=T'$.

If $S=S'$, then $aS=aS'$ and if $T=T'$ then $bT=bT'$, where a and b are supposed to be known quantities.

Then, $aS+bT=aS'+bT'$. Also if this equation and either of the given ones hold true, the remaining given one holds true.

But in simultaneous quadratic equations all kinds of changes were made in the given equations in the processes of solution. Hence the plan of verifying roots found in the original equation or equations should be regularly followed.

II. THEORY OF QUADRATIC EQUATIONS

304. Relation between the Coefficients and Roots in Quadratic Equations.

1. Every quadratic equation containing one unknown number may be reduced to the form $x^2 + px + q = 0$.

This can always be accomplished by

(a) *Clearing of fractions, when fractions appear;*

(b) *Transposing all terms to the left member;*

(c) *Collecting, and, when necessary in literal equations, factoring x^2 out of two or more x^2 terms and x out of two or more x terms.*

(d) *Dividing the equation through by the coefficient of x^2.*

Because any quadratic equation can be reduced to the form $x^2 + px + q = 0$, we will study this type of equation.

2. Solve the following eight typical equations, all except the last two by the factoring method.

(1) $x^2 - 5x + 6 = 0$. (2) $x^2 + 5x + 6 = 0$.
(3) $x^2 - x - 6 = 0$. (4) $x^2 + x - 6 = 0$.
(5) $x^2 - 4x + 4 = 0$. (6) $x^2 - 2x = 0$.
(7) $x^2 - 2x - 6 = 0$. (8) $x^2 - 2x + 6 = 0$.

3. Questions on the preceding solutions. A careful study of the solutions of the preceding equations raises the following questions:

(1) What number of roots has every quadratic?

(2) What relation exists between the coefficient of x and the sum of the two roots of the equation?

If the student has trouble answering these questions, he should study §§ 208, 209 anew.

(3) What relation exists between the known term in the equation and its two roots?

(4) If all the terms of an equation are positive and its roots are real, what can you say of the signs of the roots?

(5) If all the terms of an equation are positive except that containing x, and its roots are real, what can you say of the signs of its roots?

(6) If the known term of an equation is negative, what signs have the two roots?

(7) If the known term and the term containing x of an equation are both negative, which is numerically greater, the positive or the negative root? If the term containing x is positive instead of negative, which root is numerically greater?

(8) If the known term of an equation is absent, what value has one of the roots in every case?

(9) If the left member of the equation whose right member is zero is a perfect square, what can you say of the relative size of the two roots?

(10) If the known term is positive and greater than the square of half the coefficient of x, what will invariably be the character of the roots?

The question arises, can we *prove* the answers to the preceding questions true in general?

3. Proofs concerning the *number*, *sum*, and *product* of the roots of a quadratic equation.

To prove these theorems we will first solve the equation

$$x^2+px+q=0.$$

$$4x^2+4px=-4q. \qquad \text{(Hindu Rule.)}$$

$$4x^2+4px+p^2=p^2-4q. \qquad \text{(Completing square. Ax.?)}$$

$$2x+p=\sqrt{p^2-4q}. \qquad \text{(Ax.?)}$$

$$x=-\tfrac{1}{2}p\pm\tfrac{1}{2}\sqrt{p^2-4q}. \qquad \text{(Ax.?)}$$

(1) Taking the sign before the radical as positive, we get one root, and taking it negative we get a second root. Thus, there are *two* roots to a quadratic equation.

(2) Prove by adding the two roots just found, viz.,

$$-\tfrac{1}{2}p+\tfrac{1}{2}\sqrt{p^2-4q} \text{ and } -\tfrac{1}{2}p-\tfrac{1}{2}\sqrt{p^2-4q},$$

that the sum of the roots of any equation equals the coefficient of x in the given equation with its sign changed.

(3) Prove by multiplying one of the two roots just found by the other, that their product is q, that is, prove *that the product of the roots is always equal to the known term in the left member.*

The student may now verify that this truth holds in Exs. (7), (8) in 2 above.

a It must be borne in mind that the theorems in (2) *and* (3) *just proved hold true only when the coefficient of x^2 is unity; or when the equation has the form $x^2 + px + q = 0$.*

b. In the study of the quadratic which follows, the expression,

$$\sqrt{p^2 - 4q}, \text{ or } p^2 - 4q,$$

takes a prominent place. The quantity $p^2 - 4q$ is called the **discriminant** of the quadratic, because by it we can tell all about the roots of the equation, as to whether they are equal, rational, irrational, imaginary, or one of them is zero; also as to their signs, in certain cases.

(4) If p and q are both positive and $p^2 > 4q$, making $p^2 - 4q$ positive, then $\sqrt{p^2 - 4q}$ is real. Then both the roots are real. We see that both roots are negative; since $-\frac{1}{2}p \pm \frac{1}{2}\sqrt{p^2 - 4q}$ is negative whether the sign before the radical is + or −, because $\frac{1}{2}\sqrt{p^2 - 4q}$ is numerically less than $\frac{1}{2}p$; it is numerically less than $\frac{1}{2}p$, because if p^2 is *diminished* by $4q$ and then the square root is extracted, the result is less than p.

(5) If p alone is negative and $p^2 > 4q$, then $-\frac{1}{2}p$ is a *positive* quantity. Also, as before, $\frac{1}{2}p > \frac{1}{2}\sqrt{p^2 - 4q}$. Hence, both roots are positive.

(6) If q is negative, $p^2 - 4q$ is positive since p^2 itself is positive, and if q is negative the term $-4q$ becomes positive. Notice also that in this case $\frac{1}{2}\sqrt{p^2 - 4q} > \frac{1}{2}p$. Hence, when the sign before the radical is positive, the root is positive; and when it is negative, the root is negative.

(8) If $q = 0$, $-\frac{1}{2}p + \frac{1}{2}\sqrt{p^2 - 4q} = 0$. Hence, one root is 0.

(9) If $p^2 - 4q = 0$, or $q = \left(\frac{p}{2}\right)^2$, the two roots are equal, each

being $-\frac{1}{2}p$. Notice that when $q=\left(\frac{p}{2}\right)^2$, q "completes the square" and makes the left member a perfect square.

(10) If q is positive and $p^2 < 4q$, both roots are *imaginary*.

TABULAR DISCUSSION OF $x^2+px+q=0$

I. $p^2 > 4q$	1. $p>0$, $q>0$	Both roots negative
	2. $p<0$, $q>0$	Both roots positive
	3. $p>0$, $q<0$	Pos. root (numerically) $<$ neg. root
	4. $p<0$, $q<0$	Pos. root (numerically) $>$ neg. root
II. $p^2=4q$	1. $p>0$	Roots both negative and equal
	2. $p<0$	Roots both positive and equal
III. $p^2<4q$		Roots imaginary
IV. $p=0$	1. $q<0$	Roots equal numerically but opposite in sign
	2. $q>0$	Roots imaginary
V. $q<0$		Roots always real

305. Exercises in telling the Character of the Roots of Quadratic Equations. If the equation is not given in the form $x^2+px+q=0$, change it to that form. First apply the test $q<0$; then, if necessary, $p^2-4q>0$ test to see if the roots are real. Then tell by the coefficients the character of the roots as explained in the preceding article.

1. $x^2+22x-75=0$.
2. $5x^2+20x=25$.
3. $x^2-6x=0$.
4. $x^2-\frac{2}{3}x=32$.
5. $4x^2+15=16x$.
6. $1+9x^2=6x$.
7. $23x+5=10x^2$.
8. $3x^2+8x+3=0$.
9. $(x-1)(x-2)=1$.
10. $2x^2-11x+16=0$.
11. $6x^2-35x-6=0$.
12. $4z^2+12z+11=0$.
13. $\frac{x^2}{9}+\frac{x}{3}+\frac{35}{4}=0$.
14. $(x+4)^2=8x+25$.

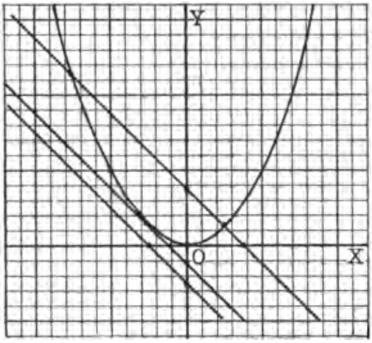
15. $\dfrac{2x+1}{1-2x}-\dfrac{5}{7}=\dfrac{x-8}{2}$.

16. $\dfrac{9}{2x+1}+\dfrac{3}{x-3}=4$.

306. Graphical Explanation of real and different, real and coincident, and imaginary roots of the quadratic.

1. Solve by the method of § 223 the three equations.

$$4x^2+4x-3=0,$$
$$4x^2+4x+1=0,$$
$$4x^2+4x+2=0.$$



In the diagram in the margin which contains the solutions of all three equations we see that the roots of $4x^2+4x-3=0$, are real and distinct; the roots of $4x^2+4x+1=0$, are real and coincident, the changing of the value of the known term causing the straight line graph to move to the left until the straight line is tangent to the curve and the two roots of the first equation have *come together*; and the roots of $4x^2+4x+2=0$, are imaginary, the straight line never meeting the parabola at all.

2. Solve by the method of § 223 the following equations, using one diagram for all:

$$x^2-x+1=0,\quad x^2-2x+1=0,\quad x^2-3x+1=0,$$
$$x^2+3x+1=0,\quad x^2+2x+1=0,\quad x^2+x+1=0.$$

Explanation. Notice in this case that all the straight line graphs go through the point (0, − 1). The first does not intersect the parabola, the second is tangent to it, giving coincident roots, the third intersects it in one point and would intersect it in another if both were extended. The other three equations give corresponding lines on the other side of the y-axis.

a. **Parameters.** The student is asked to observe that in Ex. 1, the known term changed and the other two remained the same, which had the effect of keeping the graphs parallel. In Ex. 2, by letting the coefficient of x change, and the other terms remain the same, the graphs became rays going out from (0, − 1). When one of the coefficients of an equation is allowed to change in this way it is called a **parameter**.

307. The Symbolical Forms, $\frac{0}{a}, \frac{a}{0}, \frac{0}{0}$.

1. If the numerator of a fraction changes in value and approaches indefinitely closer and closer to 0, while the denominator remains the same, or is constant, the fraction also approaches indefinitely close to 0 in value. In this way we are led to write $\frac{0}{a} \doteq 0$.

2. If the numerator of a fraction remains constant and the denominator decreases indefinitely towards 0, the value of the fraction evidently *increases* indefinitely in value. Hence with this understanding we can write

$$\frac{a}{0} \doteq \infty. \text{ (§ 9.)}$$

3. If both numerator and denominator decrease indefinitely towards 0, and we have no means of knowing their relative values, then the value of the fraction $\frac{0}{0}$ is *indeterminate.*

a. We have already seen (§ 9) that division by 0 has no meaning. But the above "forms" are convenient and are used by mathematicians. Perhaps the best way to view them is to think of 0 as denoting the *limit* towards which a variable is approaching.

Thus, $\frac{0}{5} = 0$ is to be understood as a short form of stating that the *limit* towards which $\frac{x}{5}$ tends, when x approaches the limit 0 is 0.

To show that $\frac{0}{0}$ has different values or is "indeterminate" we may write,

$$\frac{x^2-1}{x-1} = \frac{0}{0} = x + 1 = 2 \text{ (when } x = 1\text{)};$$

$$\frac{x^2-4}{x-2} = \frac{0}{0} = x + 2 = 4 \text{ (when } x = 2\text{)}.$$

308. Further Discussion of the Quadratic.

If $ax^2 + bx + c = 0$ is solved, the roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. If $c = 0$, $x = \frac{-b+b}{2a} = 0$; or $x = \frac{-b-b}{2a} = -\frac{b}{a}$.

2. If $a = 0$, and $-b \pm \sqrt{b^2 - 4ac} \neq 0$, $x = \infty$.

3. If $a = 0$, and $-b \pm \sqrt{b^2 - 4ac} = 0$, x is indeterminate.

To understand these results the symbol $\doteq$ should be used. Thus, $a \doteq 0$ means that a is an infinitesimal quantity which **approaches indefinitely close to 0.** The symbol $\doteq$ is read "approaches the limit."

Using this sign, we have:

(1) If $c \doteq 0$, $x \doteq 0$, or $x \doteq -\frac{b}{a}$.

(2) If $a \doteq 0$, and $-b + \sqrt{b^2 - 4ac} \neq 0$, $x \doteq \infty$.

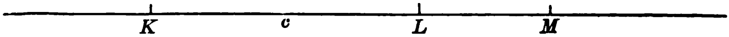
(3) If $a \doteq 0$, and $-b \pm \sqrt{b\ - 4ac} \doteq 0$, $x =$ indeterminate.

III. DISCUSSION OF PROBLEMS

309. The Courier Problem, often given in texts, is well adapted to illustrate what is meant by "the discussion of a problem." It gives rise to a simple equation.

A and B travel in the same direction at the rate of a and b miles per hour respectively. A arrives at a certain place, K, at a certain time, and B arrives at L, which is c miles from K, d hours after (*or* before) A was at K. In how many hours from the time A was at K will they be or were they together?

Solution. Let x = number of hours from time A was at K until one passes the other at M.



Then, $KM = ax$; $LM = (x - d)b$;

and $ax = c + (x - d)b$.

Solving, $ax - bx = c - bd$.

$$x = \frac{c - bd}{a - b}. \quad \textit{Ans.}$$

I. Suppose $a > b$ and $c > bd$. Then x is positive, which shows one carrier overtook the other *after* A was at K, both traveling in the direction K to M.

During the d hours before B reached L he traveled bd miles. But, by supposition, $bd < c$. Thus, when A was at K, B was between K and L. Since A travels faster than B by supposition, he will catch up with B at some place to the right of where B was when A was at K. But we do not know whether M is to the right of L as represented in the diagram or to the left of it.

We have, $LM = (x-d)b = \left(\frac{c-bd}{a-b} - d\right)b = \frac{(c-ad)b}{a-b}$.

If, now, $c > ad$, then LM is positive, and M lies to the right of L as we supposed and so represented in the diagram; but if $c < ad$, then LM is negative, and M lies to the *left* of L.

Notice, if c is supposed greater than ad, it will take A more than d hours to reach L, and he will pass B at a point to the right of L; but if c is less than ad, A will catch up with B before the d hours are up, that is, before B reaches L.

II. Suppose $a > b$, but $c < bd$. This makes the value of x negative, which shows A passed B *before* he reached K.

III. Suppose $a < b$, and $c > bd$. This makes x negative and shows that B passed A at a point to the left of K. In this case B travels more rapidly than A and will reach K before A does.

IV. Suppose $a < b$, and $c < bd$. This makes x positive and shows that B overtook A to the right of K. In this case B travels more rapidly than A and was *behind* A when he arrived at K.

V. If $c = bd$, then $x = 0$. In this case A and B are together when A is at K.

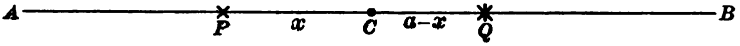
VI. If $a = b$, and $c \neq bd$, then $x = \infty$. Here A travels as fast as B and one is behind the other. The former will never catch up with the latter, or, as mathematicians say, he will catch up with him at infinity.

VII. If $a = b$, and $c = bd$, then $x = \frac{0}{0}$. In this case A and B are together at K, and, walking at the same rate, they *stay* together. Thus, x can have *any* value.

2. Prepare a corresponding discussion of this problem on the supposition the couriers A and B travel in opposite direction instead of the same direction.

3. Change the conditions of 1, by supposing that A arrives at K d hours after B arrived at L, both traveling in the same direction.

310. Clairaut's Problem of the Lights leads to the solution of a quadratic equation.



Two lights at P and Q are a feet apart. It is required to find the points in AB which are equally illuminated by the two lights. Let C be one such point, x feet from P.

By a law of optics the intensity with which a light shines at any outside point is inversely proportional to the square of the distance from the point to the source of the light.

Let $m^2 =$ illumination by light P one foot from P,

and $n^2 =$ illumination by light Q one foot from Q.

Then the illumination at C from P is $\frac{m^2}{x^2}$ and from Q is $\frac{n^2}{(a-x)^2}$.

Hence,
$$\frac{m^2}{x^2} = \frac{n^2}{(a-x)^2}.$$

$$\frac{m}{x} = \pm\frac{n}{a-x} \qquad \text{(Root Ax.)}$$

$$\therefore ma - mx = \pm nx, \text{ or } mx \pm nx = ma,$$

giving
$$x = \frac{ma}{m \pm n}.$$

1. What does the double sign in the value of x teach as regards the *number* of points of equal illumination?

2. If + in the value of x is taken, what kind of a fraction is $\frac{m}{m+n}$, proper or improper? Where then is C with reference to P and Q? Between them or on one side or the other?

3. If $-$ in the value of x is taken, and $n < m$ making the value of the fraction positive, what kind of fraction is $\frac{m}{m-n}$, proper or improper? Is the corresponding value of x greater or less than PQ? Where then will this point of equal illumination be located?

4. If $-$ in the value of x is taken, and $n > m$, what sign has value of x? In which direction from P will the corresponding point of equal illumination lie?

We see, then, from the preceding that there are always *two* points of equal illumination, one lying *between* P and Q, and the other either to the right of Q or to the left of P depending on which is the stronger light.

5. If $m = n$, $x = \frac{ma}{2m}$, or $\frac{ma}{0}$. Thus, one value of $x = \frac{1}{2}a$, locating c midway between P and Q, and the other point is at ∞.

6. If $a = 0$, and $m = n$, $x = \frac{0}{0}$, which is indeterminate. Thus, if the two lights are together and one is as strong as the other, they will illuminate *every* point equally.

IV. DEFINITIONS GENERALIZED

311. The Generalization of Definitions. In the history of mathematics it has been found over and over again that definitions and principles and proofs that were satisfactory for one stage of the development of the subject would become unsatisfactory later. The student is now in position to acquire some appreciation of this fact.

312. Definition of Addition. By adding is meant literally "putting together." But this putting together may be done in more than one way. The only addition of lines, for instance, arithmetic would recognize, would be putting the several lines together, end to end, so as to form one long line. In algebra, since there can be both positive and negative lines, the lines can be "put together" by laying one of them off *backwards* on the other. In arithmetic when we put dollars together to make a sum, the dollars must be all of one kind. In algebra two different kinds of dollars (– and +) are added or put together. Thus, "add" or "put together" has a wider meaning in algebra than in arithmetic. In higher mathematics, lines (representing *vectors* or *forces*) which do not extend in the same or opposite directions, but make an angle with each other, are "added."

313. Definition of Subtraction. In arithmetic subtraction is commonly defined as the process of *taking away* a smaller number from a larger. This is what the word itself means. But in algebra each letter is supposed to be capable of assuming any value, and so simple an expression as $a - b$ becomes *impossible* from the arithmetic standpoint if b is greater than a. One is thus led to substitute for the old definition the following: To subtract from one number another number is to find a third which added to the second of the given numbers produces the first. This definition will hold no matter what values the minuend and subtrahend may have.

314. Definition of Multiplication. The definition of multiplication often given is that multiplication is the process of taking the multiplicand as many times as there are units in the multiplier. We may picture this process by thinking of a person picking out from a basket containing eggs six at a time five times, "taking" them from one place to another. But to take them, say, $5\frac{1}{2}$ times evidently has no meaning. We must enlarge our definition, then, if fractions are to enter as multipliers. This is accomplished by saying: multiplication is the process of doing to a given multiplicand what was done to unity to get the multiplier.

For example, let us multiply 5 by $3\frac{1}{2}$. To get $3\frac{1}{2}$ from 1 we write $1+1+1+\frac{1}{2}$ of 1. Then to get $3\frac{1}{2}$ times 5, we write $5+5+5+\frac{1}{2}$ of 5, which equals $17\frac{1}{2}$. Thus $3\frac{1}{2} \times 5 = 17\frac{1}{2}$.

Unfortunately this definition is somewhat obscure and capable of being misunderstood. So far as its wording is concerned, we can get, say, 3 from 1 in this way: $1^2+1^2+1^2=3$. Then to multiply 5 by 3, we should have to add 5^2 three times, or $5^2 + 5^2 + 5^2$, which gives 75. Thus, 3×5 equals 75, which is absurd.

A much more general definition of multiplication is the following: multiplication is a process that is an extension of a particular kind of addition satisfying certain rules such as the distributive, commutative, and associative laws.* As we vary the rules we get different kinds of multiplication. Thus, there is a branch of higher mathematics called vector analysis in which the commutative law does not hold true.

315. Definition of Raising to Powers. The definition of a power which says a power of a number is the product arising from using the number as a factor one or more times fails for negative and fractional exponents.

It is convenient to distinguish between integral and fractional powers. An integral power of a number is the continued *product* arising from multiplying unity by the given number one or more times. Then, a negative integral power of a number denotes the continued *quotient* of unity divided by the given number, then this quotient divided by the given number, and so on. A number with a fractional exponent is defined as the number obtained by raising the given number to the power indicated by the numerator of its exponent and then extracting the root of the result indicated by the denominator of the fractional exponent. All this can be summed up and be made more general at the same time by saying: involution is the process subject to the following rules:

$$a^m \cdot a^n = a^{m+n};\ a^n b^n = (ab)^n;\ (a^n)^m = a^{mn}.$$

316. The Development of the Idea of Numbers. The first conception of numbers was doubtless that of integers and probably originated with the act of counting. Later, through attempting to measure objects, the idea of fractions entered. Later still, through attempting to measure certain lines, such as the diagonal of a square with one side as measuring unit, the notion of irrationals appeared. (See § 189.) This was followed long after-

* See Chrystal's *Algebra*, Part I, p. 12.

wards by the conception of still other forms of irrationals, or incommensurables, such as π.

Another extension of number was made by Argand in his diagrammatic representation of imaginaries, § 201. From what was said in § 201, it appears that any and every point in a plane is located by some complex number, and every complex number locates a point.

317. The Definition of Mathematics itself, commonly given as *the science of quantity*, has been found too narrow to include certain branches of higher mathematics. Pierce has given this definition: "Mathematics is the science which draws necessary conclusions from given premises." This definition evidently makes mathematics include logic.

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